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On the Proof Theory of Conditional Logics

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Abstract

This thesis can be ideally placed at the intersection of three research topics: conditional logics, proof theory and neighbourhood semantics. The family of logics under scope stems from the works of Stalnaker and Lewis, and extends classical propositional logic by means of a two-place modal operator, which expresses a fine-grained notion of conditionality. The semantics of these logics is modularly defined in terms of neighbourhood models.

The research aim is to investigate the proof theory of conditional logics, by defining sequent calculi for them. The proof systems introduced are extensions of Gentzen's sequent calculus; they are either *labelled*, defined by enriching the language, or *internal*, which add structural connectives to the sequents. Moreover, the calculi are standard: they are composed of a finite number of rules, each displaying a fixed number of premisses.

The thesis is organized in six chapters. Chapters 1 contains an axiomatic and semantic overview of conditional logics, while Chapter 2 is a short introduction to proof theory. The original contributions to the subject are presented in chapters 3 – 6. Chapter 3 introduces labelled calculi based on neighbourhood models for preferential conditional logics, and Chapter 4 presents different internal proof systems covering counterfactual logics, a subfamily of preferential logics. Chapter 5 analyses the relationship between proof systems by presenting a mapping between a labelled and an internal calculus. Finally, the proof-theoretic methods developed for conditional logics are applied in chapter 6 to a multi-agent epistemic logic.

Résumé

La thèse se place à l'intersection de trois sujets de recherche : logiques conditionnelles, théorie de la démonstration et sémantique de voisinage. La famille de logiques conditionnelles considérées provient dès ouvrages de Stalnaker et Lewis. Elle est une extension de la logique classique propositionnelle avec un opérateur modal à deux places, qui exprime une notion affinée de conditionnalité. La sémantique de ces logiques est définie en termes de modèles de voisinage.

Le but de la recherche est d'étudier la théorie de la démonstration des logiques conditionnelles, en précisant leur calculs des sequents. Les calculs définis sont des extensions du calcul des sequents de Gentzen ; ils sont étiquetés, c'est à dire définis en enrichissant le langage, ou internes, qui rajoutent des connecteurs structurels aux sequents.

La thèse est organisée en six chapitres. Le chapitre 1 présente les axiomes et la sémantique des logiques conditionnelles et le chapitre 2 introduit la théorie de la démonstration. Les contributions originelles au sujet sont traitées dans les chapitres 3 – 6. Le chapitre 3 introduit des calculs de sequents étiquetés basés sur la sémantique de voisinage pour les logiques conditionnelles préférentielles. Le chapitre 4 présente différents systèmes internes de calcul pour les logiques counterfactuelles, une sous-famille des logiques préférentielles. Le chapitre 5 analyse la relation parmi les systèmes de preuve en présentant les deux côtés d'une traduction entre un calcul étiqueté et un calcul interne. Finalement, au chapitre 6, les méthodes de la théorie de la démonstration conditionnelle sont appliquées à une logique épistémique multi-agente.

Yhteenveto

Tämä väitöskirja sijoittuu ideaalisesti kolmen tutkimusaiheen risteykseen: konditionaalien logiikka, todistusteoria, ja ympäristösemantiikka. Katettujen logiikoiden perhe periytyy Stalnakein ja Lewisin töistä ja laajentaa klassillista lauselogiikkaa kaksipaikkaisen modaalioperaattorin kautta. Se ilmaisee hienojakoisen konditionaalisuuden käsitteen. Näiden logiikoiden semantiikka määritellään modulaarisesti ympäristösemantiikan termein.

Tutkimuksen tavoitteena on selvittää konditionaalien logiikan todistusteoriaa määrittelemällä niille sekvenssikalkyyleita. Esitellyt todistussysteemit ovat Gentzenin sekvenssikalkyylin laajennuksia; ne ovat joko merkeillä varustettuja, joihin päädytään kielen rikastuksella, tai sisäisiä, joihin päädytään lisäämällä sekvensseihin rakenteellisia konnektiiveja. Nämä laajennukset ovat myös standardikalkyyleja: ne muodostuvat äärellisestä määrästä sääntöjä, joissa jokaisessa on kiinteä määrä premissejä.

Väitöskirja muodostuu kuudesta kappaleesta. Kappale 1 sisältää aksiomattisen ja semanttisen katsauksen konditionaalien logiikkaan, ja kappale 2 on taas lyhyt johdatus todistusteoriaan. Uudet tutkimuspanokset aiheeseen on esitetty kappaleissa 3-6. Kappale 3 esittää ns. merkityt sekvenssikalkyylit joiden perustana on preferentiaalisten konditionaalien logiikan ympäristömallit, kappale 4 esittää erilaisia sisäisiä todistussysteemejä jotka kattavat kontrafaktuaalien logiikat, jotka ovat preferentiaalisten logiikoiden alaperhe. Kappale 5 analysoi eri todistussysteemien välisiä suhteita merkkikalkyyylien ja sisäisten kalkyylien välisen kuvauksen kautta. Kuudennessa ja viimeisessä kappaleessa sovelletaan konditionaalien logiikalle kehitettyjä todistusteoreettisia metodeja useamman agentin episteemiseen logiikkaan.

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Had you rather Caesar were living and die all slaves,
than that Caesar were dead, to live all free men?

– A counterfactual from
William Shakespeare, *Julius Cæsar*

"Ford!" he said, "there's an
infinite number of monkeys outside
who want to talk to us about
this script for Hamlet they've worked out."

– Douglas Adams,
The Hitchhiker's Guide to the Galaxy

Contents

Introduction

General aim

This thesis can be ideally placed at the intersection of three research topics: conditional logics, proof theory and neighbourhood semantics.

A wide number of formal systems are classified under the name "conditional logics", and specifically all the logics apt to represent a kind of conditionality more fine-grained than classical material implication. In the present work we exclusively analyse conditional logics defined as extensions of classical propositional logic by means of a two-place modal operator which, depending on the logic under scope, can be read as a non-monotonic inference, as a counterfactual or as expressing a form of conditional belief. This approach to conditional logics, which we call the *possible worlds* account, was initially devised by Robert Stalnaker in [98] and David Lewis in [57]. Since their work these systems have been widely studied, mainly from a model-theoretic viewpoint; until recent years little was known about the proof theory of conditional logics.

Proof theory is the discipline which studies proofs within formal systems as mathematical objects (derivations), analysing their structure, properties and interactions. We are concerned with sequent calculus, a particularly rich and expressive formalism introduced by Gerhard Gentzen in [58]. Sequent calculus presents two main advantages with respect to other proof systems. First, this formalism is suitable to perform an automated proof search: the formula whose derivability is to be tested is placed at the root, and the calculus rules are mechanically applied to it, until one reaches either an initial sequent (instance of an axiom), or a sequent to which no rules are applicable. Secondly, sequent calculi allow us to represent the inference rule of *modus ponens* (MP), by means of the rule of cut:

$$\frac{\vdash A \quad \vdash A \to B}{\vdash B}\ (\text{MP}) \qquad \rightsquigarrow \qquad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

The rule of cut is problematic since it is not analytic: formula A is not a subformula of any formula in the conclusion of the rule. However, the cut-elimination theorem (one of the crucial theorems to be proved for a system of sequent calculus), ensures that the occurrences of cut can be removed from any derivation, thus preserving full analyticity.

Neighbourhood semantics is a topological semantics particularly relevant for our purposes. Neighbourhood models are built upon non-empty sets of worlds. A neighbourhood is a set of possible worlds, and the model associates to each world a set of neighbourhoods, called a "system of neighbourhoods". The worlds in the same neighbourhood can be thought of as displaying the same degree of similarity among themselves. Neighbourhood models are more expressive than Kripke models, and can be employed to define in a modular way the semantics of a wide family of conditional logics, as well as modal logics.

The aim of this thesis is to investigate the proof theory of conditional logics. In particular, we consider conditional logic $\mathbb{PCL}$ (*Preferential Conditional Logic*) and conditional logic $\mathbb{V}$ (*logic of Variably strict conditionals*). These logics, respectively introduced by Burgess in [18] and Lewis in [57], represent the basic systems of a family of conditional logics: by adding axioms, or specifying conditions on the models, it is possible to characterize a number of other logics. The extensions considered are those with normality, total reflexivity, (weak) centering, uniformity, absoluteness and combinations of them.

We present original sequent calculi for $\mathbb{PCL}$, $\mathbb{V}$ and their families. The introduced proof systems are extensions of Gentzen's propositional calculus which can be qualified as *standard*, meaning that they are composed of a finite number of rules, each displaying a fixed and finite number of premisses. Moreover, the calculi are either *labelled* or *internal*. Labelled calculi enrich the language by means of syntactic variables, the labels, representing elements of the model. These calculi have been investigated by a number of authors since the early 1990's. Here we follow the methodology of labelled contraction-free sequent calculi defined by Sara Negri in [69]. The labelled proof systems presented in the following chapters make use of the neighbourhood semantics for conditional logics [72]. Conversely, internal calculi enrich the structure of sequents by introducing additional meta-theoretic connectives. Examples of internal calculi that enrich the structure of Gentzen's systems are hypersequent calculi (defined, among others, by Arnon Avron in [9]) and nested calculi (defined, among others, by Kai Brünnler in [17]). We introduce internal proof systems for conditional logics by adding suitable structures to the formalism of sequent calculus.

Structure of the thesis and main results

The thesis is organized in six chapters. Chapter 1 and 2 are introductory: the former offers a presentation of conditional logics, with a historical survey and a discussion of relevant axioms and semantics. Chapter 2 shortly introduces proof theory, exposing the proof systems for propositional and modal logics. Specifically, Gentzen sequent calculus for propositional logics, labelled sequent calculi for modal logics and hypersequent and nested calculi for **S5** are reviewed.

Chapters 3, 4, 5 and 6 represent original contributions. Chapter 3 introduces a family of labelled sequent calculi that captures in a modular way $\mathbb{PCL}$ and its extensions, including Lewis' family of conditional logics ($\mathbb{V}$ and its extensions). The labelled calculi are defined on the basis of neighbourhood semantics. The idea to define calculi for conditional logics on the basis of neighbourhood semantics was inspired by [75], where Negri and Olivetti presented a calculus for $\mathbb{PCL}$. The correspondence between neighbourhood models and preferential models for $\mathbb{PCL}$ can be found, among others, in [63]. Based on this background, Chapter 3 extends the proof-theoretic results to all the extensions of $\mathbb{PCL}$, and gives a detailed proof of the equivalence between neighbourhood and preferential models for the whole family of logics.

Chapter 4 is devoted to internal proof systems for Lewis' logics. Three proof systems are presented, for different sub-families of logics: one-level nested sequents for $\mathbb{V}$ and some of its simplest extensions, extended hypersequents for logic $\mathbb{VTU}$ (logic $\mathbb{V}$ plus total reflexivity and uniformity) and its extensions with weak centering, centering and absoluteness, and head hypersequents for logics with total reflexivity and absoluteness. A sequent calculus for $\mathbb{V}$ was originally presented in [80]; the internal calculi of Chapter 4 extend the result to most logics in Lewis' family, being the first internal and standard proof systems to capture these logics, with varying degrees of modularity.

Chapter 5 analyses the relationship between labelled and internal proof systems for conditional logics and presents the two directions of a mapping between an internal and a labelled sequent calculus for logic $\mathbb{V}$. These translations are inspired by similar results for modal logics, as the one defined by Goré and Ramanayake in [40]. Even though it is limited to $\mathbb{V}$, the basic logic of Lewis' family, the result is not straightforward: in particular, proving the soundness of the translation from the labelled to the internal calculus requires several additional definitions and lemmas. As a reward, we obtain a clear semantic interpretation of the internal sequents in neighbourhood models.

Finally, Chapter 6 studies logic $\mathbb{V}$ extended with total reflexivity and absoluteness ($\mathbb{VTA}$), a system of conditional logic which, in its multi-agent version, is suitable to express the epistemic notion of static conditional belief. The logic corresponds to system $\mathbb{CDL}$ (*Conditional-Doxastic Logic*) treated by Alexandru Baltag and Sonja Smets in [11, 13, 12]. For this logic, a labelled sequent calculus is introduced, again based on neighbourhood models. Up to now, this proof system is the only sequent calculus known for $\mathbb{CDL}$. Moreover, the semantics for the logic of conditional belief is usually defined in terms of epistemic plausibility models; Chapter 6 presents a direct proof of completeness of the axiomatization of $\mathbb{CDL}$ with respect to neighbourhood semantics. Thus, the chapter contains a application of the proof-theoretic methods developed for conditional logics in the field of epistemic logic.

A part of the results presented in the thesis have already been published. Here follows the list of articles in which they appear.

- Chapter 3 - The labelled sequent calculus for $\mathbb{V}$ can be found in [39].
- Chapter 4 - The nested sequent calculus for $\mathbb{V}$ and its extensions can be found in [35]. The hypersequent calculi for $\mathbb{VTU}$ and extensions can be found in [37].
- Chapter 5 - The equivalence result was published in [39].
- Chapter 6 - The labelled calculus for $\mathbb{CDL}$ was first published in [35]; a more comprehensive discussion can be found in [38].

Chapter 1

Conditional logics: a gentle introduction

I believe there are possible worlds other
than the one we happen to inhabit.
– Davis Lewis, *Counterfactuals*

There are different ways to describe a logic, once its language is fixed. A *syntactic* characterization consists in specifying a set of axioms and inference rules which allow to derive true propositions from the axioms. A *semantic* characterization amounts to the definition of an adequate class of models for the logic: a model is a structure that, under a given interpretation, assigns the right truth values to compound formulas of the language. Theorems of *soundness* and *completeness* relate the syntactic with the semantic characterization, proving that a formula can be derived from the axioms if and only if it is valid in the model.

In this chapter we introduce conditional logics, generally intended as formal systems suitable to treat a fine-grained notion of conditionality. We then proceed to isolate the class of logics that are going to be the subject of the following chapters: conditional systems which extend classical propositional logic by means of a two-place modal operator. We first give a semantic characterisation, presenting the classes of models for the logics, and then a syntactic one, presenting the axioms and inference rules of the systems.

1.1 Conditionals in natural language

The term "conditionals" denotes a wide class of natural language sentences. They can be defined as expressions relating a proposition (the antecedent) to another proposition (the consequent). Moreover, in the English language conditionals are usually expressed as sentences of the form:

If A then B.

The use of conditionals in natural language is extremely wide and differentiated: distinct *if ... then* clauses give rise to different kinds of conditionals. Some relevant examples are factual conditionals, which assess some cause-effect relation in the world ("If I strike the match, it will light"); non-normal conditionals, expressing a relation that is true in most cases ("If Tweety is a bird, it can normally fly"), conditionals expressing obligation

("If you break the neighbour's window, you ought to tell him"), epistemic conditionals assessing beliefs or knowledge of an agent ("If Albert knows that Beatrice wears a white hat, he knows he is wearing a red one") and counterfactual conditionals, expressing consequences of a state of affairs that did not obtain: "If kangaroos had no tail, they would topple over" [57]. These latter can be qualified as conditional sentences whose antecedent is always false in the actual world.

This list is not exhaustive; there have been some attempt to classify conditionals (or at least, to distinguish them in some categories), all leading to problematic results. The most famous distinction is the one between *indicative* and *subjunctive* conditionals: in the English language, conditionals at the present (indicative) tense usually express factual conditionals, while conditionals in the subjunctive tense express counterfactuals. However, some counterexample can be found:

If he has solved the problem, I'm the Queen of England.

The meaning of this conditional is counterfactual, but its form is indicative [61]. Moreover, any classification of conditional sentences is language-specific: different languages might have different ways to express conditionals.

We shall stick to the observation that there are different kinds of conditional sentences, generally defined as propositions in the form *if... then* and expressing a link between a proposition (the antecedent) and another proposition (the consequent). The question is whether and how conditionals can be accommodated within a formal system of logic.

1.2 Truth-functional implication

Classical logic interprets conditionals in terms of the operator of *material implication*, here denoted with $\to$. Material implication is truth-functional, meaning that the truth-value of the compound statement $A \to B$ is determined as a function of the truth values of its single components: A, the antecedent, and B, the consequent. This yields the well-known truth table for material implication,

A	B	$A \to B$
1	1	1
1	0	0
0	1	1
0	0	1

according to which implication can be defined through the disjunction

$$(A \to B) \leftrightarrow \neg A \vee B.$$

Thus, a conditional sentence is classically true if either its antecedent is false or its consequent is true. This interpretation gives rise to paradoxical consequences: relevance of the premisses and conclusion is not taken into account, and we end up validating as true some intuitively problematic sentences, such as "If 2+2=5, New York is the capital of New Zeland", corresponding to *ex falso quodlibet.* Similarly, each conditional sentence with true consequent is true. These cases are known as "paradoxes of material implication", and might concern also factual conditionals, as the one above.

Furthermore, material implication is not fine-grained enough to capture the different truth values we might intuitively assign to counterfactuals. In the classical framework, counterfactuals are conditional sentences whose antecedent is false, and thus are all considered to be true. For instance, these two statements would have the same truth value:

(1) If Clinton had won the elections, Trump wouldn't be president.
(2) If Clinton had won the elections, she would have refused the position.

We might want to differentiate between true and false counterfactuals: (1) should be evaluated as true, and (2) should be evaluated as false, at least if we assume Clinton's willingness to become president.

Several solutions have been proposed to give an account of conditionals more faithful to such distinctions. Usually, different logics are specifically tailored to capture some kinds of conditionals; up to now, no formal system has been devised that is able to accommodate all kinds of conditionals. Relevance logic have been proposed to impose a degree of *relevance* between antecedent and consequents; and non-monotonic logics use a non monotonic inference relation, allowing to draw defeaisble conclusions that can be changed in light of some new informations: so, if I learn that Tweety is a hen, I will correctly conclude that it does not fly. As for counterfactuals, two main approaches have been devised: the possible worlds account (Stalnaker, [98] and Lewis, [57]) and the probabilistic account (Adams, [1, 2]). According to the probabilistic account, conditionals are not truth-value bearers: instead of truth values, they can be assigned degrees of acceptability. More precisely, to each proposition A is associated a probability $P(A)$; the probability of a conditional *if* A *then* B is given as the conditional probability $P(B|A)$ of its components:

$$P(B|A) = \frac{P(A\&B)}{P(A)}$$

Adams formulated his theory to formalize indicative conditionals, and the application of the probabilistic account to counterfactuals is problematic: the antecedent of counterfactuals should have probability zero, since the antecedent describes a state of affairs that did not obtain, and to calculate the conditional probability of a counterfactual we should divide by zero. Lewis strongly criticized Adams' system and, following Stalnaker, proposed a completely different theory of conditionals, which we call the *possible worlds account*. According to the possible worlds account, conditionals have truth-values; they can be formalised by means of a two-place modal operator in the context of Kripke models for modal logics. The thesis is focused on the possible worlds account; thus, we analyse conditional logics as extensions of classical propositional logic by means of a two-place modal operator.

For a survey on modal logics refer to Appendix 1.A at the end of this chapter. Here we report some notational conventions.

Definition 1.2.1. Given a Kripke model $\mathcal{M}$, we denote validity (or truth) of a formula A at world x either by $x \in [\![A]\!]$ or by $x \Vdash A$. We denote validity (or truth) of A at a model as $\mathcal{M} \vDash A$, meaning that the formula is true at all the worlds in the model. We denote validity of A as $\vDash A$, meaning that the formula is valid in all models. In this case, we say that A is a *theorem* of the modal logic.

Remark 1.2.1. A truth-functional operator can be classified as "extensional", in opposition to "intensional" operators. Intensional operators are operators whose truth conditions do not depend exclusively on the truth values of the formula components. In this sense, a modal operator is intensional: to evaluate the truth value of $\Box A$ at some world x we need to evaluate A at the different worlds of the model which are related to x. Due to the failure of truth-functional implication to capture counterfactuals, it comes as a natural solution to resort to an intensional operator, such as a modal operator.

1.3 Possible worlds account of conditionals

The possible worlds account of counterfactuals is due to Stalnaker in [98] and Lewis in [57]. This approach can be generalized to other conditionals, as the non-monotonic and the deontic conditional. Important contributions to the possible worlds account of conditionals can be found in Chellas in [19] and Nute in [77].

This account relies on the assumption that conditionals are truth-value bearers. C.I. Lewis suggested in [55] to interpret counterfactuals by means of the *strict conditional*, that is, a one-place modal operator preceding a classical implication. We use $>$ to denote a (counterfactual) conditional.

$$A > B = \Box(A \to B).$$

In Kripke models the truth condition for this formula would be:

$$x \Vdash \Box(A \to B) \text{ if for all } y \in W \text{ such that } xRy \text{ it holds that } y \nVdash A \text{ or } y \Vdash B.$$

This interpretation works fine to capture simple counterfactual sentences such as "If Alice came, it would have been a great party"[1]. Let A stand for "If Alice came", and P for "It would have been a great party". To evaluate the counterfactual $A > P$, we have to check whether in the possible worlds accessible from the actual world it holds that, if A is true, also P is true. In Figure 1.1 there are two Kripke models; the small circles represents worlds, and formulas written inside them are formulas true at that world. The arrow connecting two worlds represents the accessibility relation. We evaluate the formula at world x, the actual world.

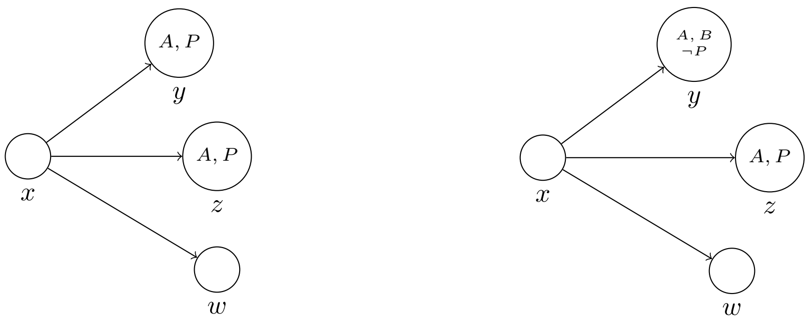


Figure 1.1: The party example

In the left-hand side model, the counterfactual $A > P$ is valid: in all the hypothetical situations in which A is true, P holds as well.

Now consider the following proposition: "If both Alice and Bob came, it wouldn't have been a great party", which we represent by the strict conditional $\Box(A \wedge B \to \neg P)$. Intuitively, the formulas $A > P$ and $(A \wedge B) > \neg P$ could be simultaneously true: suppose Alice is really good at animating parties, unless when her ex-boyfriend Bob is around. However, we cannot devise a model in which both formulas $\Box(A \to P)$ and $\Box(A \wedge B \to \neg P)$ are true at x and, simultaneously, at x is true formula $\Diamond(A \wedge B)$. Consider the right-hand side model of Figure 1.1. If there is at most one world in which both A and B are true (in this case, y), any assignment of truth-values to formulas at the worlds fails to satisfy either $\Box(A \to P)$ or $\Box(A \wedge B \to \neg P)$.

[1]This example is know as the "party example", and it is quite famous in the literature. It was originally proposed in [57].

Stalnaker's and Lewis' solution consists in introducing a two-place modal operator to represent counterfactuals. The Kripke semantics has to be enriched to accommodate this new operator. Basically, the semantics establishes an ordering among worlds so that different worlds are used to evaluate different conditionals. Stalnaker employs a function to select the worlds relevant for one conditional; Lewis defines layers of accessible worlds around the actual world. This solution allows to overcome the difficulties of the strict conditional, and gives a quite fine-grained account of counterfactuals.

1.4 A semantic account

In this section we present the principal models devised for conditional logics in the possible worlds account: selection function models, preferential models, sphere models and neighbourhood models. Selection function models, due to Stalnaker, were the first models introduced specifically tailored to capture the counterfactual conditional. Preferential models and sphere models are both due to Lewis. Neighbourhood models are a more general class of models, originally introduced for non-normal modal logics[2]. Recall that, even if these classes of models were initially devised to capture counterfactuals, they can be adapted also to other conditionals: in particular, preferential models can be used to represent some systems of non-monotonic logics.

Definition 1.4.1. The set of well-formed formulas $\mathcal{F}$ of a propositional conditional logic is generated as follows, for p atomic formula and A, B formulas:

$$\mathcal{F} \ ::= \ P \mid \bot \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid A > B$$

The connective $>$ is the (counterfactual) conditional operator. We shall define its truth conditions in the different systems of semantics.

Selection and class-selection function models

Stalnaker in [98] defines selection function models: a class of models that enriches reflexive and transitive Kripke models by means of a selection function, f. This function takes as arguments a world x and, for a formula A, the set of worlds in which A is true, denoted by $[\![A]\!]$. The image of the function is the world y most similar to x and in which A holds:

$$f(x, [\![A]\!]) = y.$$

The selection function establishes an ordering among words with respect to their similarity. The truth condition of a counterfactual conditional at selection function models is the following:

$$x \Vdash A > B \quad \textit{iff} \quad f(x, [\![A]\!]) \Vdash B$$

Stalnaker explained what it means to evaluate a counterfactual within these models by means of the following "thought experiment"[3]:

[2]Non-normal modal logics fail to satisfy either the axiom of distributivity or the rule of necessitation. Refer to Appendix 1.A.

[3]Stalnaker claims to have been inspired by what he calls the "Ramsey test", which is actually a footnote in Ramsey's article [93] in which Ramsey describes the process of evaluating a conditional: *If two people are arguing "If p, then q?" and are both in doubt as to p, they are adding p hypothetically to their stock of knowledge and arguing on that basis about q; so that in a sense "If p, q" and "If p, q" are contradictories. We can say that they are fixing their degree of belief in q given p. If p turns out false, these degrees of belief are rendered void. If either party believes not p for certain, the question ceases to mean anything to him except as a question about what follows from certain laws or hypotheses.* However, Stalnaker's reading of the Ramsey test is not exactly accurate: Salnaker assigns truth condition to counterfactuals, while Ramsey does not.

> First, add the antecedent (hypothetically) to your stock of beliefs; second, make whatever adjustments are required to maintain consistency (without modifying the hypothetical belief in the antecedent), finally, consider whether or not the consequent is then true. [98]

Thus, to evaluate a counterfactual we first choose the world y such that $f(x, [\![A]\!]) = y$, i.e., the world most similar to x (the actual world) in which A holds (since we are evaluating counterfactuals, A might not hold in the current world). Then, we verify whether in y the consequent holds.
By means of example, suppose we want to evaluate the following counterfactual:

> *If Facebook hadn't sold its data to Cambridge Analytica, Trump wouldn't have won the elections.*

To evaluate this counterfactual, we select the world most similar to the current world in which the antecedent is true i.e., one in which Facebook has not sold any data, and see whether in this world Trump has won the elections or not.

We shall not formally define Stalnaker's selection function models[4]. For our purpose, it suffices to say that Lewis criticized Stalnaker's models, due to the fact that they rely on two fundamental (unjustified, according to Lewis) assumptions:

- the *Limit assumption*, stating that we can always find at least one world closest to the actual one.
- the *Stalnaker assumption*, stating that there is exactly one world closest to the actual one.

The Limit assumption corresponds to the requirement that the similarity ordering among worlds is a well-founded preorder, with no infinite descending chains. This assumption always holds in finite models, and might be ignored if the logic has the finite model property. Stalnaker's assumption, on the other hand, defines one of the strongest systems of conditional logics: it states that we can define an ordering of worlds with respect to their similarity such that there is always exactly one world that best resembles a certain world. This assumption can be easily criticized, on the grounds that there could be more than one world similar to the same degree to the actual one.

Chellas and Nute also studied selection function models; we adopt the definition of models given by Nute in [77]. In this class of models, called *class-selection function models*, the selection function chooses a set of accessible possible worlds with the same degree of similarity with respect to the actual world. This generalization allows to capture a wide number of conditional logics weaker than Stalnaker's system.

Definition 1.4.2. A *class-selection function model* is a structure $\mathcal{M} = \langle W, f, [\![\ ]\!] \rangle$, where:

- W is a non-empty set of elements;
- $f : W \times \mathcal{P}(W) \rightarrow \mathcal{P}(W)$ is the selection function;
- $[\![\ ]\!]$ is the propositional evaluation.

The selection function chooses the worlds $f(x, [\![A]\!])$ most similar to x given the information A.

$x \Vdash A > B$ iff B is true in all worlds selected by $f(x, [\![A]\!])$.

More recently, these models have been studied in [5], along with the proof theory of the corresponding logics. Class-selection function models can be extended by adding

[4] The interest reader might refer to Stalnaker's original article [98].

conditions on the selection function, in the same way as modal logic **K** can be extended by adding conditions on the accessibility relation. For instance, if we add the condition $f(w, [\![A]\!]) \subseteq [\![A]\!]$ we obtain a model in which the world selected by f must satisfy formula A.

Preferential models

Lewis introduced preferential models in [57], presenting them as an equivalent alternative to sphere models (refer to next section). Burgess in [18] and Halpern and Friedman in [28] studied preferential models (and the corresponding logics) on their own right. The definitions in this section are from [75].

Preferential models specify a binary accessibility relation to be added to a Kripke model. The accessibility relation imposes an ordering among the worlds of the model, which can be interpreted as expressing either the comparative similarity among worlds or the normality of a world with respect to the other. In any case, the relation chooses a set of accessible worlds which are "preferred" with respect to the actual one. Thus, to each world x is associated a set of accessible worlds and a *comparative similarity* relation $y \leq_x z$ on the accessible worlds, meaning y is at least as similar, or plausible, as z with respect to x.

Definition 1.4.3. A *preferential model* is a structure

$$M = \langle W, \{W_x\}_{x \in W}, \{\preceq_x\}_{x \in W}, [\![\ ]\!] \rangle$$

composed of the following elements: a non-empty set of worlds, W; a propositional evaluation for atomic formulas, $[\![\ ]\!]$; for every world $x \in W$, a set W_x of worlds accessible from it, and a binary relation $\preceq_x$ over W_x. The relation $\preceq_x$ satisfies the following properties:

- *Reflexivity*: For all $w \in W$ $w \preceq_x w$;
- *Transitivity*: For all $w, k, z \in W$ if $w \preceq_x k$ and $k \preceq_x z$ then $w \preceq_x z$.

The truth condition for the conditional operator is the following:

$$\begin{gathered} x \Vdash A > B \quad \textit{iff} \quad \textit{for all } w \in W_x, \textit{ if } w \Vdash A \textit{ then} \\ \textit{there exists } y \preceq_x w \textit{ such that } y \Vdash A \textit{ and for all } z \preceq_x y, z \Vdash A \rightarrow B. \end{gathered}$$

Intuitively, for a formula $A > B$ to be true at world x it must hold that, for all worlds in W_x, if there is a world w in which A is true, then there is a world y at least as similar as w to x such that y satisfies A and all worlds at least as similar as y to x satisfy $A \rightarrow B$.

Remark 1.4.1. The truth condition takes into account the fact that there could not be worlds minimal with respect to $\preceq_x$ which satisfy A. If we assume the Limit assumption, the model has no infinite descending chains of worlds; and thus, there is a minimal world with respect to $\preceq_x$. In this case (or in finite models, where the Limit assumption holds) the truth condition can be simplified as follows:

$$x \Vdash A > B \ \textit{ iff } \ \textit{for all } w \in W_x \textit{ if } w \in \mathsf{Min}_x(A) \textit{ then } w \Vdash B$$

where

$$\mathsf{Min}_x(A) = \{y \in W_x \mid y \Vdash A \textit{ and for all } z \in W_x \textit{ if } z \preceq_x y \textit{ and } z \Vdash A \textit{ then } y \preceq_x z\ \}.$$

Thus, in finite models, to evaluate a conditional $A > B$ at x means to check whether the classical implication holds in the worlds minimal with respect to the relation $\preceq_x$, since B must be true at the worlds minimal with respect to $\preceq_x$ and in which A holds.

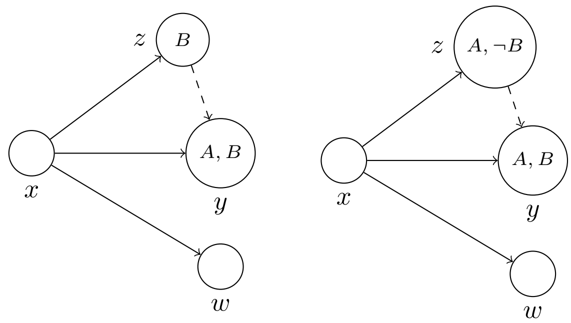


Figure 1.2: Preferential models

By means of example, consider the models in Figure 1.2 in which the arrow signifies the accessibility relation (thus, $z, y, w \in W_x$) and the dashed arrow stands for the preference relation (thus, $z \preceq_x y$). World x in the model on the left satisfies the conditional $A > B$: it holds that $y \in W_x$ and $y \Vdash A$. Moreover, since $y \preceq_x y$ (the model is reflexive) we verify on y the consequent of the truth condition. It holds that for all $k \preceq_x y$, $k \Vdash A \to B$: the worlds accessible from y are y itself and z, and both satisfy B. In the right-hand side model, on the contrary, world x does not satisfy the conditional: world z satisfies A and $\neg B$, and the consequent of the truth condition fails.

According to Lewis, preferential models are not suited to represent counterfactuals [57]. The problem is that preferential models do not necessarily require that all worlds in W_x to be related by the preference relation. For instance, consider world w in Figure 1.2: we do not know how this world is related to the others by the preference relation, and thus is incomparable. Preferential models are better suited for a non-monotonic interpretation, where $A > B$ is read as "usually, if A then B". In this case, the preference relation between worlds is interpreted as expressing a notion of normality between worlds: a sentence "usually, birds can fly" is considered true if the classical implication is verified in the worlds closest, or normal, with respect to the actual worlds.

Preferential models do not require all worlds in W_x to be related by the preference relation. We can impose this requirement adding the condition of *connectedness* to $\preceq_x$, to the class of models.

Definition 1.4.4. A *connected preferential model* is a preferential model such that the relation $\preceq_x$ satisfies also the property of

Strong connectedness: For all $w, k \in W_x$ it holds that $w \preceq_x k$ or $k \preceq_x w$.

In connected models $\preceq_x$ is a *total preorder* (a transitive and connected relation) on W_x, this meaning that all elements of W_x are comparable one with the other. Connected preferential models are particularly significant since they can be proved to be equivalent to Lewis' sphere models.

Sphere models

Sphere models were introduced by Lewis in his seminal work Counterfactuals [57]. These models generalise the structure of Kripke models introducing *spheres*, sets of worlds sharing the same degree of similarity to the actual world. Lewis explained his idea referring to C.I. Lewis' interpretation of counterfactuals: counterfactuals are not "strict conditionals based on comparative similarity", but moreover "*variably* strict conditionals based on comparative similarity" [57].

Lewis defined a whole family of sphere models for conditional logics. To simplify the exposition, we first define the simplest class model of the family, and then define the class of *centered* sphere models. Centered sphere models are important since, according to Lewis, these are the models better suited to formalize counterfactuals.

Definition 1.4.5. A *sphere model* is a structure

$$\mathcal{M} = \langle W, S, [\![\ ]\!] \rangle$$

composed of a non-empty set of elements, W; a function $S : W \to \mathcal{P}(\mathcal{P}(W))$, associating to each world x a set of sets of worlds $S(x)$ and $[\![\ ]\!]$, the propositional evaluation. Moreover, S satisfies the following properties:

- *Non-emptiness*: For each $\alpha \in S(x)$, α is non-empty;
- *Nesting*: For each $\alpha, \beta \in S(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$;
- *Closure under non-empty union*: If $H \subseteq S(x)$ and $H \neq \emptyset$, then $\bigcup H \in S(x)$;
- *Closure under non-empty intersection*: If $H \subseteq S(x)$ and $H \neq \emptyset$, then $\bigcap H \in S(x)$.

Remark 1.4.2. Lewis' original properties defined in [57, Chapter 1] are slightly different: he assumes closure under union, from which follows that the empty set is a sphere around each world, $\emptyset \subseteq S(x)$. From this follows that $\bigcup \emptyset \in S(x)$, and $\emptyset \in S(x)$. Lewis admits that the inclusion of the empty set is "unintuitive", and accepted for technical convenience. We have chosen to include at its place the condition of non-emptiness. Moreover, the two conditions of closure under non-empty union and intersection always hold in finite models, and Lewis proves that his systems have the finite model property. Thus, the soundness and completeness results hold also in the absence of these conditions [57, Chapter 6, footnote 1]. This is the reason why we do not consider these properties when dealing with the proof theory of Lewis' system.

Definition 1.4.6. A *centered sphere model* is a sphere model in which S additionally satisfies the property of

Centering: For each $\alpha \in S(x)$ it holds that $\{x\} \in S(x)$ so that $\{x\} \subseteq \alpha$.

We introduce the following notation, from [72], to treat satisfiability of a formula at a sphere.

Definition 1.4.7. Given a sphere $\alpha \subseteq W$ we write:

$\alpha \Vdash^{\exists} A$ if there exists $x \in \alpha$ such that $x \Vdash A$;
$\alpha \Vdash^{\forall} A$ if for all $x \in \alpha$, $x \Vdash A$.

Thus, $\alpha \Vdash^{\exists} A$ means that there is at least a world in α in which A is true, and $\alpha \Vdash^{\forall} A$ that A is true at all worlds in α.

Definition 1.4.8. Employing the above notation, the truth condition for the conditional operator in sphere models is the following:

$x \Vdash A > B$ *iff* *if there exists* $\alpha \in S(x)$ *such that* $\alpha \Vdash^{\exists} A$*, then there exists* $\beta \in S(x)$ *such that* $\beta \Vdash^{\exists} A$ *and* $\beta \Vdash^{\forall} A \to B$.

Figure 1.3 shows a centered sphere model. Spheres around x represent degrees of similarity with respect to the world x: the closer the sphere to x, the more similar the worlds in the spheres are to x. The union on all the spheres is the set of worlds accessible from the actual world x (thus, w is not "seen" from the actual world). World x is placed at the centre of the system, into the smallest sphere containing no other worlds. The reason is that the actual world is the world most similar to itself, and thus the smallest

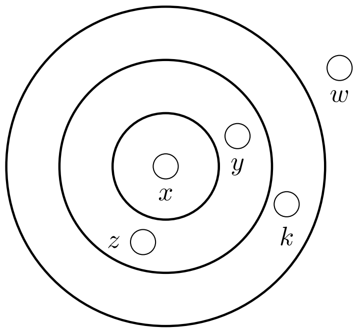


Figure 1.3: Centered sphere model

sphere around it has to be a sphere containing no other worlds. *Nesting* of the system of spheres is required to compare the different worlds belonging to the sphere in terms of similarity.
For a counterfactual conditional $A > B$ to be true at world x, it holds that, if there is some world in some sphere that satisfies the antecedent, then there exists a sphere in which at least one world satisfies the antecedent, and all worlds in the sphere satisfy the classical implication $A \rightarrow B$. This is equivalent to the following requirement: for a counterfactual to be true ata world x, the world closest to x in which A is true must satisfy also B; or, there should not be a world satisfying A and $\neg B$ closer to x than any world satisfying A and B. The intuition is the same as in Stalnaker's model: to evaluate a counterfactual, we evaluate whether the consequent holds at the closest possible worlds at which the antecedent holds.

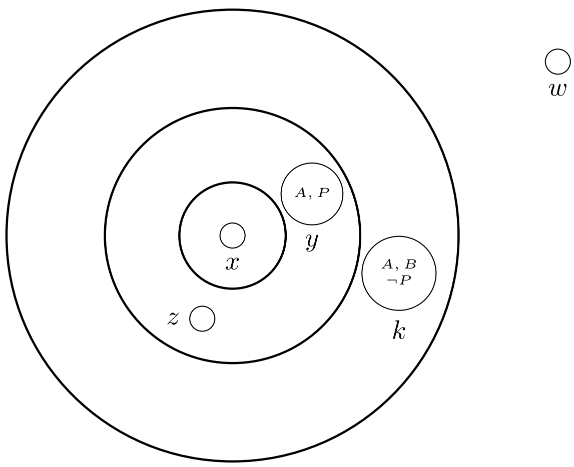


Figure 1.4: Party example II

Figure 1.4 shows a centered sphere model in which world x verifies $\Diamond(A \wedge B)$ and both counterfactuals $A > P$ and $A \wedge B > \neg P$ from the previous example regarding Anna's presence at the party (Figure 1.1). In this model, both sentences are true, but at different spheres. This is the reason why Lewis calls his counterfactuals *variably* strict conditionals: differently from what happened with C.I. Lewis' strict conditionals, the set of worlds used to evaluate each conditional might vary, becoming more or less strict.

Centered sphere models are the models Lewis considered most suitable to formalize counterfactuals. In [57] he proposed a wide number of sphere models, accommodating different forms of conditionality.

Definition 1.4.9. Extensions of sphere models are defined by adding the following properties to the function S (centering, already mentioned, is also present in the list):

- N *Normality*: For all $x \in W$ it holds that $S(x) \neq \emptyset$;
- T *Total reflexivity*: For all $x \in W$ there exists $\alpha \in S(x)$ such that $x \in \alpha$;
- W *Weak centering*: For all $x \in W$, for all $\alpha \in S(x)$ it holds that $x \in \alpha$;
- C *(Strong) Centering*: For all $x \in W$, for all $\alpha \in S(x)$ it holds that $\{x\} \in S(x)$, so that $\{x\} \subseteq \alpha$;
- U *Local uniformity*: For all $x \in W$ and $y \in \alpha$ for $\alpha \in S(x)$ it holds that $\bigcup S(x) = \bigcup S(y)$;
- A *Local absoluteness*: For all $x \in W$ and $y \in \alpha$ for $\alpha \in S(x)$ it holds that $S(x) = S(y)$;
- L *Limit assumption*: for some formula A, if there is some sphere $\alpha \in S(x)$ such that $\alpha \Vdash^{\exists} A$ there is a smallest sphere $\beta \in S(x)$ such that $\beta \Vdash^{\exists} A$;
- All *Universality*: for all $x \in W$ it holds that $\bigcup S(x) = W$.

Remark 1.4.3. Some of the conditions are incremental: C implies W, W implies T; T implies N; A implies U and U + T are equivalent to All and St implies L. Moreover, a model containing W + A is equivalent to a model for **S5**, meaning that all the spheres collapse into one, while a model containing C + A collapses to a model for classical logic (in which all the worlds collapse to one).

The basic system of sphere models is too weak to represent any counterfactual[5]. As for conditions weaker than C, Lewis motivates W admitting that the smallest sphere of the model might be composed of the actual world plus several other worlds, all similar to the actual one[6]. Thus, W requires only that the actual world belongs to *all* the spheres. Condition T is weaker, since the actual world might not be at the centre of the system of spheres. According to Lewis, sphere models with T are not suitable to represent counterfactuals, but could be useful to represent deontic conditionality. In a deontic sphere model the ordering relation among worlds can be interpreted as expressing moral improvement: thus, worlds belonging to inner spheres are morally better than worlds belonging to outer sphere, and worlds belonging to the innermost sphere are the morally perfect ones. Since the actual world is not necessarily perfect from a moral viewpoint, it might belong to some outer sphere. Condition N only requires (together with non-emptiness) the system of spheres to be non-empty, and might be apt to represent some weak form of deontic conditional.

Local uniformity and absoluteness impose restrictions on the spheres: for local uniformity, the set of worlds belonging to all the spheres is always the same, while the stronger condition of local absoluteness requires the system of spheres to be the same for all worlds[7]. Models featuring U, A or a combination of them are indicated by Lewis as suitable for deontic interpretations.

In Section 1.5 we detail the axioms corresponding to the above properties of the models, along with theorems of soundness and completeness.

[5] According to Lewis, this system is apt to capture only *egocentric* logic. This system is quite peculiar; refer to [57, Chapter 5] for its description.

[6] In such a model, we have that counterfactuals with true antecedents and true consequents are not necessarily true: we have to verify truth of the antecedent not only at the actual world, but also at the actual world closest / most similar to the actual one.

[7] The conditions of local uniformity and absoluteness can be equivalently stated as *global* conditions: *Uniformity* states that for all $x, y \in W$ it holds that $\bigcup S(x) = \bigcup S(y)$, and *Absoluteness* that for all $x, y \in W$ it holds that $S(x) = S(y)$. Lewis proves that systems with uniformity or local uniformity and absoluteness or local absoluteness satisfy the same set of formulas [57, Chapter 6]. We use the global version of these conditions in Chapter 4.

Neighbourhood models

Neighbourhood models were introduced by Scott in [96] and Montague in [67] to give an interpretation of non-normal modal logics; they are a generalization of sphere models, in which to each world is associated a (not necessarily nested) set of worlds, used to interpret modalities. Grove employed these models in constructing a system of semantics for belief revision [42]; more recently, neighbourhood models were studied by Pacuit in [84], Marti and Pinosio in [63], Negri and Olivetti in [75] and Negri in [72].

Definition 1.4.10. A *neighbourhood model* is a structure

$$\mathcal{M} = \langle W, N, [\![\ ]\!] \rangle$$

where W is a non empty set (of possible worlds); N is the *neighbourhood function* $N : W \to \mathcal{P}(\mathcal{P}(W))$ and $[\![\ ]\!] : Atm \to \mathcal{P}(W)$ is the propositional evaluation.
The neighbourhood function associates to each $x \in W$ a set $N(x) \subseteq \mathcal{P}(W)$, called *neighbourhood*. We denote by Greek letters $\alpha, \beta, ...$ elements of $N(x)$. For all $x \in W$, we assume the neighbourhood function to satisfy the property of *non-emptiness*: For each $\alpha \in N(x)$, α is non-empty.

Truth conditions for Boolean combinations of formulas are the standard ones; for the conditional operator we have:

$$x \Vdash A > B \quad \textit{iff} \quad \textit{for all } \alpha \in N(x) \textit{ it holds that if } \alpha \Vdash^{\exists} A \textit{ then there exists } \beta \in N(x) \textit{ such that } \beta \subseteq \alpha,\ \beta \Vdash^{\exists} A \textit{ and } \beta \Vdash^{\forall} A \to B.$$

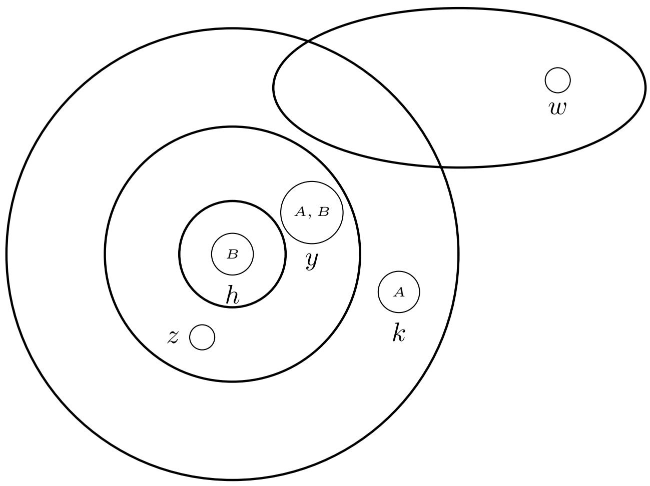


Figure 1.5: Neighbourhood model

Figure 1.5 shows a neighbourhood model in which a conditional $A > B$ is satisfied at world x. In the figure the system of neighbourhood associated to x is shown.

Lewis noted that sphere models are a special kind of neighbourhood models, without the nesting condition [57]. As with preferential models, neighbourhood systems are not suited to represent counterfactuals, but some weaker form of conditionality (such as prototypical interpretation). Thus, Lewis rejected neighbourhood systems; for us, on the contrary, they will be extremely useful since we employ them to uniformly capture both logics weaker than Lewis' system and with Lewis' system, adding the condition of nesting.

1.5 An axiomatic account

The different classes of models shown in the previous section are tailored to capture specific systems of logic. Figure 1.6 presents a set of axiom schemata and inference rules characterizing these logics.

We define the relation of derivability of a formula A in an axiom system as follows: $\vdash A$ if and only if formula A is derived from the axioms using the rules of inference.

$\mathbb{CK}$		Axiomatization of classical propositional logic
	(RCEA)	$\dfrac{A \leftrightarrow B}{(A > C) \leftrightarrow (B > C)}$
	(RCK)	$\dfrac{A \rightarrow B}{(C > A) \rightarrow (C > B)}$
	(R-And)	$(A > B) \wedge (A > C) \rightarrow (A > (B \wedge C))$
$\mathbb{PCL}$		Axiomatization of $\mathbb{CK}$
	(ID)	$A > A$
	(CM)	$(A > B) \wedge (A > C) \rightarrow ((A \wedge B) > C)$
	(RT)	$(A > B) \wedge ((A \wedge B) > C) \rightarrow (A > C)$
	(OR)	$(A > C) \wedge (B > C) \rightarrow ((A \vee B) > C)$
$\mathbb{V}$		Axiomatization of $\mathbb{PCL}$
	(CV)	$((A > C) \wedge \neg(A > \neg B)) \rightarrow ((A \wedge B) > C)$

Figure 1.6: Axiom systems of conditional logics

Remark 1.5.1. Axioms (CM) and (RT) can be replaced by axiom

$$\text{(CSO)} \quad ((A > B) \wedge (B > A)) \rightarrow ((A > C) \leftrightarrow (B > C)).$$

$\mathbb{CK}$ is the smallest normal conditional logic, where a normal conditional logic is defined as a logic closed under (RCEA) and (RCK) [19]. The logic was first presented by Chellas in [19], and it is sound and complete with respect to class-selection function models (refer to [19] for the proof). The notion of conditionality that $\mathbb{CK}$ expresses is very weak, and for this reason this logic can be regarded as the "basic" conditional system, meaning that successive theories - both of deontic or of counterfactual conditionals - include the notion of conditionality defined by $\mathbb{CK}$. Thus, the logic is not of particular interest in itself; but it plays the same role as system **K** of modal logics.

If we add to the axioms of $\mathbb{CK}$ the axioms of identity (ID), cumulative monotony (CM), restricted transitivity (RT) and the axiom of disjunction of antecedents (OR) we obtain an axiomatization for the logic $\mathbb{PCL}$. $\mathbb{PCL}$ stands for Preferential Conditional Logic; this system, introduced in [18], is sound and complete with respect to preferential models (refer to [18] for the proof). This logic is suitable for representing non-monotonic conditionals: more precisely, the non-monotonic logic **P** introduced by Kraus Lehman and Magidor in [47] is equivalent to the flat fragment of $\mathbb{PCL}$, i.e., the fragment in which we do not allow nesting of conditional operators to occur inside formulas. Logic **P** captures prototypical sentences; however, instead of a binary intensional connective, the system employs a non-monotonic inference relation, i.e., a binary relation symbol which is part of the meta-language. As the authors remark, the semantics of **P** is quite different from the semantics of $\mathbb{PCL}$: the truth condition for $\mathbb{PCL}$ makes reference to an actual world, which is not needed to evaluate whether the non monotonic inference relation holds between two propositions. Logic $\mathbb{PCL}$ (and its extensions) are richer system, of which **P** (and its extensions) correspond to the flat fragment. It is anyhow interesting to remark the similarity at the axiomatic level.

Finally, Lewis' basic logic $\mathbb{V}$ can be obtained by adding to $\mathbb{PCL}$ the axiom (CV) of commutativity, or strengthening of the antecedent, which allows to add formulas in the antecedent of counterfactuals[8]. Logic $\mathbb{V}$ is sound and complete with respect to sphere models (refer to [57] for the proof). As we have already seen, system $\mathbb{V}$ is, according to Lewis, too weak to capture counterfactuals; but we can add axioms to $\mathbb{V}$ and obtain a wide variety of stronger systems. In the previous section we have seen how we could add semantic conditions to sphere models; here we present axioms that can be added to $\mathbb{V}$. To each axiom there corresponds the semantic condition that goes under the same name; Lewis proves soundness and completeness for all the systems w.r.t. the sphere model enriched with the corresponding semantic condition [57].

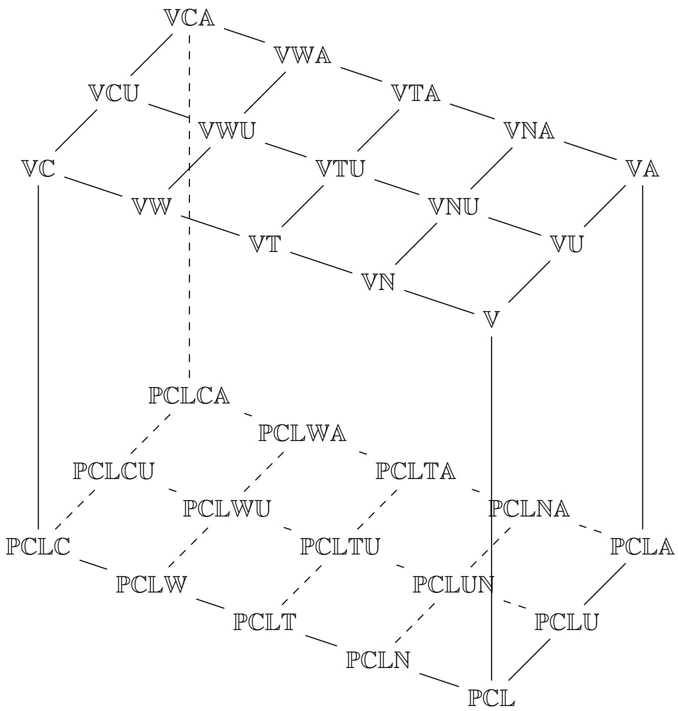


Figure 1.7: The conditional logic cube

Both local uniformity and absoluteness have a pair of characteristic axioms; universality is given by axiom (U) + (T). These axioms were introduced by Lewis, who applied them to $\mathbb{V}$ [57]. However, the same axioms can also be applied to $\mathbb{PCL}$, and generate a family of non-nested systems. Figure 1.7 and Figure 1.8 represent the cube of conditional logics extending $\mathbb{PCL}$, which will be object of the following chapters. In the figure the interdependencies between the logics are represented, following the convention that when two system are connected by a line, upper systems are extensions of the lower ones.

Remark 1.5.2. As with frame conditions, we have some interrelations among the axioms: in $\mathbb{VC}$, (C) implies (W); in $\mathbb{VW}$ (W) implies (T) and in $\mathbb{VT}$ (T) implies (N).

[8]This axiom corresponds to the semantic condition of nesting in sphere models: the strengthening of the antecedent can be applied thanks to the nested structure.

(N)	$\neg(\top > \bot)$	*Normality*
(T)	$A \to \neg(A > \bot)$	*Total reflexivity*
(W)	$(A > B) \to (A \to B)$	*Weak centering*
(C)	$(A \wedge B) \to (A > B)$	*Strong centering*
(U_1)	$(\neg A > \bot) \to \neg(\neg A > \bot) > \bot$	*Uniformity (1)*
(U_2)	$\neg(A > \bot) \to ((A > \bot) > \bot)$	*Uniformity (2)*
(A_1)	$(A > B) \to (C > (A > B))$	*Absoluteness (1)*
(A_2)	$\neg(A > B) \to (C > \neg(A > B))$	*Absoluteness (2)*

Figure 1.8: Axioms for extensions

Similarly, in $\mathbb{VA}$ both (U_1) and (U_2) are valid. Logic $\mathbb{VWA}$ collapses to $\mathbb{S}5$; logic $\mathbb{VCA}$ collapses to classical logic.

1.6 Conditional logics and modal logics

It is possible to define the modal operators $\Box$ and $\Diamond$ by means of the conditional operator.

$$\Box A := \neg A > \bot;$$
$$\Diamond A := \neg(A > \bot).$$

The modal operators defined in $\mathbb{PCL}$ and its extensions have the properties of the modal operators of normal modal logics. We detail the modal correspondence, from [57], in the case of logic $\mathbb{V}$ and its extensions; however, the result can be extended to $\mathbb{PCL}$ and its extensions.

Lewis motivates the definition of $\Box A$ as follows. By definition of the conditional operator $>$ in sphere models, a formula $A > \bot$ can be either false or vacuously true, i.e., with false antecedent. Thus, if formula $\neg A$ is either false or vacuously true, we have that formula A is non-vacuously true: A is necessary. Similarly, if it does not hold that A is false or vacuously true, it might be that A is non-vacuously true: thus, that A is possible, $\Diamond A$.

The truth condition for $\Box A$ in sphere models corresponds to the truth condition of formula $\neg A > \bot$. Recall the truth condition of the conditional operator in sphere models:

$$x \Vdash A > B \ \textit{ iff } \ \textit{if there exists } \alpha \in S(x) \textit{ such that } \alpha \Vdash^{\exists} A, \textit{ then there exists } \beta \in S(x) \textit{ such that } \beta \Vdash^{\exists} A \textit{ and } \beta \Vdash^{\forall} A \to B.$$

Thus, $x \Vdash \neg A > \bot$ iff, if there exists $\alpha \in S(x)$ such that $\alpha \Vdash^{\exists} \neg A$, then there exists $\beta \in S(x)$ such that $\beta \Vdash^{\exists} \neg A$ and $\beta \Vdash^{\forall} \neg A \to \bot$. Since the consequent cannot hold, the antecedent must be false. Thus,

$$(*)\, x \Vdash \Box A \ \textit{ iff } \ \textit{for all } \alpha \in S(x) \textit{ it holds that } \alpha \Vdash^{\forall} A.$$

Similarly, the truth condition for $\Diamond A$ in sphere models corresponds to the truth condition of formula $\neg(A > \bot)$. By definition, $x \Vdash \neg(A > \bot)$ iff there exists $\alpha \in S(x)$ such that $\alpha \Vdash^{\exists} A$ and for all $\beta \in S(x)$ it holds that $\beta \Vdash^{\forall} \neg A$ or $\beta \Vdash^{\exists} A \wedge \top$. Thus:

$$(**)\, x \Vdash \Diamond A \ \textit{ iff } \ \textit{there exists } \alpha \in S(x) \textit{ such that } \alpha \Vdash^{\exists} A.$$

The conditions $(*)$ and $(**)$ correspond to the truth condition for for $\Box$- and $\Diamond$-formulas in Kripke models. Moreover, given a system of spheres associated to world x, $(*)$ and $(**)$ can be formulated in terms of the outer sphere $\bigcup S(x)$:

$$x \Vdash \Box A \quad \textit{iff} \quad \textit{for all } w \in \bigcup S(x) \textit{ it holds that } w \Vdash A;$$
$$x \Vdash \Diamond A \quad \textit{iff} \quad \textit{there exists } w \in \bigcup S(x) \textit{ such that } w \Vdash A.$$

Thus, to interpret formulas $\Box$ and $\Diamond$ we assign to the actual world x a single sphere of accessibility: $\bigcup S(x)$, which is the outermost sphere of $S(x)$. For this reason, Lewis calls $\Box$ and $\Diamond$ the "outer" modalities. For each conditional logic he identifies its "outer" modal logic: the modal logic obtained when we interpret formulas $\neg A > \bot$ in the outer sphere of a system of spheres.

Lewis' conditional logic	*Outer modal logic*
$\mathbb{V}$	**K**
$\mathbb{VN}$	**D**
$\mathbb{VT}$, $\mathbb{VW}$, $\mathbb{VC}$	**T**
$\mathbb{VU}$, $\mathbb{VA}$	**K45**
$\mathbb{VNU}$, $\mathbb{VNA}$	**D45**
$\mathbb{VTU}$, $\mathbb{VWU}$, $\mathbb{VCU}$, $\mathbb{VTA}$, $\mathbb{VWA}$	**S5**
$\mathbb{VCA}$	Classical logic

Figure 1.9: Modal correspondences

Lewis proves that, under the definition of modal operators by means of the conditional operator, the axioms of the outer modal logic are theorems (in some cases axioms) of the corresponding conditional logic, and vice versa [57, chapter 6]. More precisely, let $\mathcal{L}$ be a logic belonging to Lewis' family, and let $\mathcal{L}^{\Box}$ be the modal fragment obtained from $\mathcal{L}$ by restricting conditional formulas to $\neg A > \bot$. Given A formula of $\mathcal{L}$, let $A^{\Box}$ be the formula obtained by replacing every subformula $\neg B > \bot$ with $\Box B$. Let **M** denote one of the modal logics listed in Figure 1.9. Then, it is possible to prove the following:

- If $A^{\Box} \in \mathbf{M}$, then $A \in \mathcal{L}^{\Box}$;
- If $A \in \mathcal{L}^{\Box}$, then $A^{\Box} \in \mathbf{M}$.

A full semantic proof can be found in [57, chapter 6]. The proof amounts to showing that, given a world $x \in W$, the sphere $\bigcup S(x)$ of the conditional logic satisfies the same properties as the set of worlds accessible from x in the outer modal logic.

System **K** is the outer modal logic of $\mathbb{V}$, since classes of models for both logics only require W to be not empty. Logic **D** is associated to $\mathbb{VN}$ since models for $\mathbb{VN}$ require normality, i.e., the presence of at least one sphere $\alpha \in S(x)$. By non-emptiness, α contains at least one element: thus, there exists $y \in \bigcup S(x)$. This corresponds to the requirement of *seriality* of the accessibility relation in models for **D**: there exists at least one element $y \in W$ such that xRy.

Logic **T** is the outer modal logic associated to $\mathbb{VT}$, $\mathbb{VW}$ and $\mathbb{VC}$ since in sphere models for all these logics it holds that $x \in \bigcup S(x)$, corresponding to the condition of *reflexivity* of the accessibility relation for **T**: for all $x \in W$, xRx. Modal logic **T** is not fine-grained enough to express the difference between total reflexivity, weak centering and centering, which can be appreciated only at the level of spheres.

As for **K45**, this system is the outer modal logic of both $\mathbb{VU}$ and $\mathbb{VA}$. Again, the difference between the conditions of uniformity and absoluteness can be stated only in terms of spheres. In sphere models for both logics, it holds that $\bigcup S(x) = \bigcup S(y)$; thus, the worlds composing the systems of spheres are all the same (possibly arranged in different spheres, in case of uniformity). Thus, since formulas $\neg A > \bot$ and $\neg(A > \bot)$ are evaluated at $\bigcup S(x)$, it holds that these formulas are uniformly true at all the worlds in $\bigcup S(x)$. The same happens in models for **K45**: if $\Box A$ is true at some world y such that xRy, for x actual world, then $\Box A$ is true at all the worlds that are accessible form x, or from some world in turn accessible from x. If we add total reflexivity (equivalently: weak centering or centering) to a sphere model for $\mathbb{VU}$ or $\mathbb{VA}$, the outer modal logic becomes **S5**: all formulas $\Box A$ are the same between all the worlds in $\bigcup S(x)$, which now includes x as well.

Models for $\mathbb{VWA}$ are composed of just one sphere, since they have to accommodate the requirement of absoluteness (all the spheres are the same) and weak centering (the actual world is included in all the spheres). However, the outer modal logic is still **S5**, since all the worlds in W, possibly more than one, occur in the sphere. Models for $\mathbb{VCA}$ are composed of just one sphere, that contains exactly one world, since for the condition of strong centering the sphere $\{x\}$ should be contained is one of the (all equal) spheres. This is the reason why this systems collapses to classical propositional logic.

Remark 1.6.1. Other than the outer modal operator, Lewis defines also a couple of *inner* modal operators:

$$\Box^i A := \Diamond A \wedge \top > A;$$
$$\Diamond^i A := \Diamond A \rightarrow \top > A;$$

where the $\Diamond A$ in the definitions is needed to rule out the possibility of an empty set of spheres. The truth conditions for the inner modal operators in sphere models are the following:

$x \Vdash \Box^i A$ *iff there exists* $\alpha \in S(x)$ *such that* $\alpha \Vdash^{\forall} A$;
$x \Vdash \Diamond^i A$ *iff for all* $\alpha \in S(x)$ *it holds that* $\alpha \Vdash^{\exists} A$.

Lewis defines inner modal logics in correspondence to each system of conditional logic in [57, Chapter 6]. Interestingly enough, the inner modal logics for $\mathbb{V}$, $\mathbb{VU}$ and $\mathbb{VA}$ are non-normal modal logics.

1.7 Problems of the possible worlds account of conditionals

The possible worlds account of conditional logics is not free from criticism; on the contrary, many problems have been found in Stalnaker-Lewis' approach to conditionals. The critics mainly address Lewis' book *Counterfactuals* and regard both the foundations of the approach and the technical aspects. The first criticisms were addressed to Lewis by Fine in [24] and Nute in [77]. Here we report some of Fine's main observations.

As Lewis recognises, the foundations of his theory are the possible worlds analysis and the relation of comparative similarity between worlds. In Chapter 4 of Counterfactuals he defends both points. Regarding possible worlds, Lewis maintains a position of realism, according to which possible worlds actually exist:

> I believe there are possible worlds other than the one we happen to inhabit. [...] I therefore believe in the existence of entities that might be called "ways things could have been". I prefer to call them "possible worlds". [57, p. 84]

Lewis defends this position saying that he does not know any convincing counterargument. However, this position leads to problems concerning trans-world identity. The last section of Chapter 1 of [57] explains how it is possible to evaluate the same individual at different worlds. For instance, to establish the truth value of a counterfactual "If Oswald hadn't shot Kennedy, someone else would have" we have to evaluate the behaviour of the individual “Oswald” at different worlds. But how can we denote “Oswald” at different worlds? Lewis’ solution is the introduction of the relation of *counterpart*: each individual exists in exactly one world, and in other worlds there are counterparts of that individual, where a counterpart for an individual is something that resembles the individual closely enough (“in important respects of intrinsic quality and extrinsic relations”, p. 39) and it resembles it no less closely than other things existing at a world. Lewis maintains that the counterpart relation explains trans-world identity, and possibly it is so; however, a simpler solution would be to reject realism of possible worlds, in favour of a position requiring a lighter ontological commitment: nominalism.

Nominalism identifies possible worlds with the set of propositions true at it - more precisely, with the “maximal collections of propositions X such that it is possible that all of the propositions in X are true”[24]. Thus, possible worlds are logical constructions, and the counterpart relation is not needed to explain trans-world identity: it is not problematic to maintain that the same proposition (say regarding a certain individual) is true at different possible worlds. In any case, Fine observes that realism about possible worlds is not so fundamental in Lewis’ theory: it does not make a difference whether we regard possible worlds as existing or as linguistic entities.

The relation of comparative similarity on which Lewis bases his account is far more problematic. According to Fine, the notion of comparative similarity among worlds gives rise to a danger of circularity, due to its “multi-criterial vagueness”. Lewis claims that the comparative similarity relation should be vague, and that this feature is the key strength of his approach.

> Counterfactuals are notoriously vague. [. . .] [Thus, an account of their truth values] must be made relative to some parameter that is fixed only within rough limits on any given occasion of language use. It is to be hoped that this imperfectly fixed parameter is familiar one [. . .]. It will be a relation of comparative similarity. [57, p. 1]

The problem is that there can be several criteria of similarity and, moreover, that the notion of similarity seems to accommodate the counterfactual we are judging: the chosen sense of similarity is the one that will yield the correct truth conditions. The question is whether this “accommodation” might be characterized without circularity. Fine concedes that a more refined similarity criterion could work, but that it is not clear to him whether the similarity relation needs to make reference to some external theory of counterfactuals.

To show how the evaluation of a counterfactual relies on comparative plausibility, and how its vagueness might be problematic, Fine considers the following counterfactual:

If Nixon had pressed the button there would have been a nuclear holocaust.

This counterfactual is evaluated as true on the basis of the existence of a world (k in the leftmost model of Figure 1.10) in which Nixon presses the button (A) and there is a nuclear holocaust (B), and there are no worlds closer to the actual world in which A and $\neg B$ are true. However, it is easy to imagine a possible world (y in the rightmost model of Figure 1.10) in which Nixon presses the button and due to a technical problem the atomic bomb is not released. Clearly, this world will be closer to the actual world than y, since a world with an electronic problem and no atomic holocaust is more similar to the actual world than a world with an atomic holocaust. Thus, the counterfactual is no longer true, since the world closest to the actual is a world in which A and $\neg B$ hold.

This can be generalised to all counterfactuals: it is easy to imagine a world with some small change that falsifies a counterfactual. This is the extent to which truth of a counterfactual relies on the comparative similarity relation, which apparently avoids all definitions due to its vagueness.

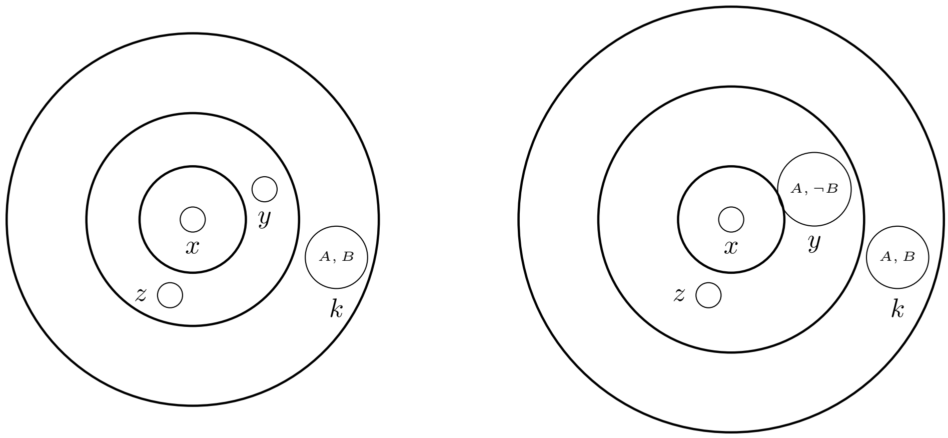


Figure 1.10: Nixon's counterexample

Many other criticisms have been addressed to Lewis' and Stalnaker approach, and many alternative theories of counterfactuals and conditionals have been devised (refer, for instance, to [77]). Even if the philosophical foundations of the theory are not as strong as one might hope, we are presently interested in a study of the formal systems (defined as axioms and inference rules) and formal models of conditional logics. We can abstract away from the intuitive interpretations of the systems, knowing that each logic is a kind of approximation of natural language - thus, imperfect. From now on, the focus will be on the proof theory of conditional logics.

1.A Appendix: Modal logics

A modal is an expression used to qualify truth of a judgement, like "necessarily", or "possibly". Thus, strictly speaking, modal logic is the logic that studies the behaviour of expressions as "it is necessary that" and "it is possible that" [30]. These two "modals" correspond to the alethic interpretations of the modal operators, respectively $\Box$ and $\Diamond$. The two operators are interdefinable:

$$\begin{aligned} \Box A &= \neg\Diamond\neg A \\ \Diamond A &= \neg\Box\neg A \end{aligned}$$

Thus, the language of a propositional modal logic **M** can be defined taking as primitive only one of these operators (we choose $\Box$):

Definition 1.A.1. The set of formulas $\mathcal{F}^{\Box}$ of a propositional modal system **M** is generated as follows, for p atomic formula and $A, B \in \mathcal{F}^{\Box}$:

$$\mathcal{F}^{\Box} ::= p \mid \neg A \mid A \wedge B \mid A \vee B \mid A \to B \mid \Box A$$

We maintain that $\wedge$ and $\vee$ bind stronger than $\to$, and omit parentheses where possible.

There are many different systems of modal logic. We present an axiomatization of the main *normal* systems[9]. These logics, represented in Figure 1.A.1 are generated by adding to modal logic **K** the axioms from the set {(D), (T), (B), (4), (5)}. Some modal logics have different axiomatizations: for instance, **S5** = **K** + {(T), (B), (4)} and **S5** = **K** + {(T), (5)}. We denote by **M** any of the modal logic represented in the modal logic cube.

We define the relation of derivability of a formula A in the axiom system for **M** as follows: $\vdash A$ if and only if formula A is derived from the axiom schemata using the rules of inference.

The systems are defined in an incremental way: **K**, the basic system, is the weaker of them, and it is included in all the others. Conversely, **S5** is the strongest system: it includes all the other systems.

The modal operators of $\Box$ and $\Diamond$ can be interpreted as expressing attitudes other than necessity and possibility (alethic interpretation), such as obligation and permission (deontic interpretation), knowledge and belief (epistemic interpretation), or necessity at any moment of time and possibility at some point in time (temporal interpretation). Some systems of modal logics are better suited to formalise certain interpretation of the modal operators: for instance, **K**, the basic system, is suited for the alethic reading; the deontic reading is best captured by **D**, and the epistemic reading by **S4** and **S5**.

Modal logics were introduced by C.I. Lewis in [56], but reached their full development with the introduction of possible worlds models, or Kripke models, that offer a clear and intuitive semantics for the modal operators.

Definition 1.A.2. A *possible worlds model* for **K** is a structure $\mathcal{M} = \langle W, R, [\![\ ]\!]\rangle$, where:

- W is a non-empty set of elements, called "worlds";
- $R \subseteq W \times W$ is a binary relation among worlds, called "accessibility relation";
- $[\![\ ]\!] : Atm \to \mathcal{P}(W)$ is a function associating to each atomic formula P the set of worlds in which P is true.

To define models for the extensions of **K** we add conditions on the accessibility relation in correspondence to axioms:

[9] A *normal* system for modal logic is a system in which both the distributivity axiom and the necessitation rule are valid.

Modal logic **K**	(0)	Any axiomatization of classical propositional logic
	(Nec)	If A is a theorem of **M**, then so is $\Box A$
	(Dist)	$\Box(A \to B) \to (\Box A \to \Box B)$

Extensions of **K**	(D)	$\Box A \to \Diamond A$
	(T)	$\Box A \to A$
	(B)	$A \to \Box\Diamond A$
	(4)	$\Box A \to \Box\Box A$
	(5)	$\Diamond A \to \Box\Diamond A$

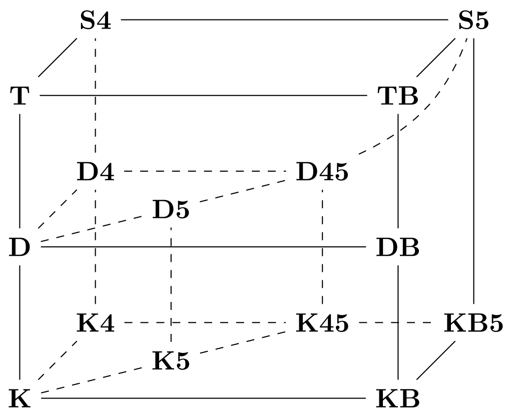


Figure 1.A.1: The modal logic cube [31]

d	*For all $x \in W$ there exists $y \in W$ such that xRy*	*Seriality*
t	*For all $x \in W$ it holds that xRx*	*Reflexivity*
b	*For all $x, y \in W$ if xRy then yRx*	*Symmetry*
4	*For all $x, y, z \in W$ if xRy and yRz then xRz*	*Transitivity*
5	*For all $x, y, z \in W$ if xRy and xRz then yRz*	*Euclideaness*

Thus, for insance, a model for **S5** is a model whose accessibility relation is euclidean and reflexive, which amounts to an equivalence relation.

Remark 1.A.1. Usually, the parts W and R of a model $\mathcal{M}$ are defined as a *frame*, i.e. a structure $\langle W, R\rangle$ to which the evaluation function $[\![\ ]\!]$ is added to obtain the full model. To define models for different modal logics we have to specify additional conditions on R; thus, the *frame* of the model is what changes between different modal logics. For this reason, the conditions to be added to R are usually called *frame conditions*.

The function $[\![\ ]\!]$ can be extended to compound formulas as follows, for $\mathcal{M}$ model for any of the above logics.

- $[\![\bot]\!] = \varnothing$
- $x \in [\![\neg A]\!]$ if $x \notin [\![A]\!]$
- $x \in [\![A \wedge B]\!]$ if $x \in [\![A]\!] \cap [\![B]\!]$
- $x \in [\![A \vee B]\!]$ if $x \in [\![A]\!] \cup [\![B]\!]$
- $x \in [\![A \to B]\!]$ if $x \in [\![A]\!] \subseteq [\![B]\!]$
- $x \in [\![\Box A]\!]$ if for all $y \in W$ such that xRy it holds that $y \in [\![A]\!]$
- $x \in [\![\Diamond A]\!]$ if for some $y \in W$ such that xRy it holds that $y \in [\![A]\!]$

Definition 1.A.3. Given a formula A of the language, and a model $\mathcal{M}$ for some modal logic $\mathbf{M}$, A is *valid (or true) at a world* x if for some $x \in W$, $x \in [\![A]\!]$. Then, A is *valid (or true) at a model* $\mathcal{M}$ if for all $x \in W$, $x \in [\![A]\!]$; $\mathcal{M}$ is a *countermodel* for A if for some world x, $x \notin [\![A]\!]$. Finally, A is *valid* if A is valid in every sphere model.

Notation 1.A.1. Instead of $[\![\ ]\!]$ we also use the forcing relation $x \Vdash A$ to denote truth of a formula A at a world x. Thus $x \Vdash A$ iff $x \in [\![A]\!]$.

Employing this notation, the truth conditions of $\Box$ and $\Diamond$ at possible worlds models become the following:

$x \Vdash \Box A$ *iff* *for all* $y \in W$ *such that* xRy, $y \Vdash A$;
$x \Vdash \Diamond A$ *iff* *there exists* $y \in W$ *such that* xRy *and* $y \Vdash A$.

Proofs of soundness and completeness relate the axiom systems with the models for each modal logic $\mathbf{M}$.

Looking at the truth condition for $\Box A$, it can be observed that the $\Box$ operator cannot be captured by means of a truth table. To evaluate truth of a formula $\Box A$ it does not suffice to take into account the truth value of A; we have to consider the truth value of A at *all* worlds in the model. Conversely, classical implication is truth-functional: truth of a formula $A \to B$ at a world x depends on truth of the formula A and B at x. The modal operators are not truth-functional, and are usually referred to as *intensional* operator.

Chapter 2

Proof methods for modal and conditional logics

There is some order in the apparent confusion.
– Anne S. Troelstra & Helmut Schwichtenberg,
Basic proof theory

A *formal system* consists of a set of axioms and inference rules, expressed in a symbolic language. A proof for a proposition in a formal system is a deductive argument (a *formal proof*, or *derivation*) that derives true propositions from the axioms, using the inference rules. Thus, a formal system can be seen as a *proof system* that allows us to find proofs for propositions.

Proof theory studies derivations as mathematical objects: it analyses their structure, properties and interactions. It is possible to formulate different proof systems for a logic, some better suited than others for a structural analysis. Structural proof theory is a branch of the discipline started by Gerhard Gentzen with the introduction of natural deduction and sequent calculus.

In this chapter we introduce structural proof theory and present sequent calculus for classical propositional logic. We then introduce some proof systems for modal logics and, finally, survey the main results regarding the proof theory of conditional logics.

2.1 Structural proof theory

Proof theory was born at the beginning of the XX century, as a product of the increasing interest of mathematicians and logicians in formal systems. David Hilbert developed the so-called *axiomatic proof theory*, which captures the formal notion of derivability within axiomatic systems. These formal systems are composed of a typically large number of axioms (more precisely: axioms schemata) and usually just one inference rule: *modus ponens*. In Chapter 1 we presented Hilbert systems for conditional logics. Formal proofs (derivations) within an Hilbert system are usually presented as a list: a derivation for formula A is a finite sequence of formulas $\mathcal{D} = A_1, \ldots, A_n$ in which each formula is either an instance of an axiom or derived from the previous formulas by means of inference rules, and such that $A = A_n$. For instance, the Hilbert system for minimal implication logic is composed of the following axioms and inference rule [102]:

(MP)	If $\vdash A$ and $\vdash A \to B$ then $\vdash B$
(k-axiom)	$A \to (B \to A)$
(s-axiom)	$(A \to (B \to)) \to ((A \to B) \to (A \to C))$

Here follows a derivation of the identity axiom $A \to A$ in the system:

(1)	$(A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A))$	(s-axiom)
(2)	$A \to ((A \to A) \to A)$	(k-axiom)
(3)	$(A \to (A \to A)) \to (A \to A)$	(MP) to (1), (2)
(4)	$A \to (A \to A)$	(k-axiom)
(5)	$A \to A$	(MP) to (3), (4)

Formal proofs within axiomatic systems can be qualified as *synthetic*: to build a derivation, one proceeds from the hypotheses / general truths (the axioms) until the conclusion (formula to be proved) is reached. Axiomatic derivations might be quite difficult to perform, since the choice of axiom schemata or inference rule to apply seems to be *ad hoc*, and is not guided by the operational meaning of the connectives of the formulas.

Gerhard Gentzen developed in his thesis [32] a different paradigm of derivability, which goes under the name of *structural proof theory.* He defined formal systems for classical and intuitionistic logic composed of few simple axioms and many inference rules, more structured with respect to the rules of an axiomatic system. Gentzen's aim was to define calculi mimicking actual mathematical reasoning: he wanted to represent as closely as possible the "actual logical reasoning involved in mathematical proofs" [32]. He devised two systems: natural deduction and sequent calculus[1].

From comparison to Hilbert's systems both Gentzen's systems can be qualified as *analytic* [102], meaning that, to construct a derivation for a formula, the formula is analysed into its simple components until an axiom is reached. In this procedure, the inference rules reflect the operational meaning of the connectives: the rules identify the simpler components needed to introduce a formula with a certain main connective, along with the subformulas that can be generated from it. Moreover, instead of a linear format, derivations in Gentzen's systems have the form of a tree: in the case of natural deduction, derivations are labelled trees of formulas, and in the case of sequent calculus they are trees labelled by sequents. The tree form allows an immediate and clear perception of the derivation structure.

By means of an example, in next section we shall present a system of sequent calculus for classical propositional logic; the general notions presented for this case can be applied to all systems of sequent calculus.

2.2 The sequent calculus G3cp

The sequent calculus **G3cp** for classical propositional logic closely resembles Gentzen's original sequent calculus in [32], and was introduced in [102].

For p atomic formula, A and B formulas of the language, the set of formulas of classical propositional logic is generated by the following grammar: $p \mid \bot \mid \neg A \mid A \wedge B \mid A \vee B \mid A \to B$. To define a system of sequent calculus we introduce the notion of sequent, specify its interpretation in the language of the logic and provide the set of initial sequents and inference rules[2].

Definition 2.2.1. A **G3cp** sequent is an expression of the form

$$\Gamma \Rightarrow \Delta$$

[1]For a treatment of natural deduction, refer to [74]. We shall here treat only sequent calculus.

[2]As we will see in next sections, the notion of sequent might be enriched to define calculi for other logics, as modal logics.

where Γ and Δ are finite multisets[3] of formulas. Γ is called the *antecedent* of the sequent, Δ the *consequent* of the sequent. Both antecedent and consequent can possibly be empty or constituted of just one formula occurrence. A sequent $\Gamma \Rightarrow \Delta$ is interpreted in the language of classical propositional logic as follows:

$$(\Gamma \Rightarrow \Delta)^{int} = \bigwedge \Gamma \rightarrow \bigvee \Delta.$$

The symbol $\Rightarrow$ represents in the object language the metalinguistic notion of consequence relation, describing how some formulas (the assumptions) entail some other formulas (the conclusions). The comma in the antecedent represents conjunction at the meta-level, while the comma in the consequent represents disjunction. Thus, a sequent can be read as: "from the conjunction of formulas in the antecedent follows the disjunction of formulas in the consequent".

An empty antecedent is a conjunction of empty formulas (which conventionally stands for the truth $\top$) and an empty consequent is a disjunction of empty formulas (which conventionally stands for a generic contradiction $\bot$). A sequent with both antecedent and consequent empty stands for a generic contradiction ($\top \rightarrow \bot$).

Initial sequents

$$\frac{}{p, \Gamma \Rightarrow \Delta, p}\ \mathsf{init} \qquad \frac{}{\bot, \Gamma \Rightarrow \Delta}\ \bot_{\mathsf{L}}$$

Logical rules

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ \mathsf{R}\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ \mathsf{R}\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\rightarrow \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ \mathsf{R}\rightarrow$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\neg \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\ \mathsf{R}\neg$$

Structural rules

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

Figure 2.1: Sequent calculus **G3cp**

Figure 2.1 presents the (sequent) rules of **G3cp**. Following [64], we define a *sequent rule instance* as an ordered pair consisting of a sequent S, called the *conclusion*, and a finite set of sequents $S_1, \dots, S_n$, called the *premisses*. A rule instance is usually written as:

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

Then, a *sequent rule* is defined as the infinite set of all its rule instances. A rule in which $n = 0$ is an initial sequent. In Figure 2.1 (and throughout the thesis) we write sequent rules by means of *rule schemata*: formulas with variables replaced by meta-variables for formulas $A, B, C \dots$, and meta-variables for finite multisets of formulas, $\Gamma, \Delta, \Sigma \dots$.

[3] A multiset is a set in which the order of the elements is not taken into account: thus, two multisets $\{A, B\}$ and $\{B, A\}$ are considered as equal.

Sequent rules can be classified as *structural*, meaning that the rule does not affect the connectives of the formulas, or *logical*, meaning that the rule modifies the connectives of at least one formula. In each logical rule we distinguish:

- the *main formula*: the formula occurring in the conclusion and containing the introduced logical symbol;
- the *secondary formulas*, or *active formulas*: formulas occurring in the premiss(es) of the inference rule which are subformulas of the main formula in the conclusion;
- the *context*: formulas occurring in the premiss(es) and the conclusion, which are untouched by the inference rule.

In the cut rule we call the *cut formula* the formula A; all the other formulas constitute the *context* of the rule.

Looking at the rules *top-down* (from the premisses to the conclusion), logical rules introduce a formula with a certain main connective either on the right or on the left of the sequent arrow. Looking at the rules *bottom-up* (from the conclusion to the premisses) logical rules analyse a formula, placed either on the left or right of the sequent arrow, according to the operational meaning of its main connective, which for classical logic is described by the truth tables. The formula interpretation of a sequent is the key to understand the meaning of the rules. Rule $\mathsf{L}\wedge$ read bottom-up substitutes $\wedge$ with the meta-connective " ,". Rule $\mathsf{R}\wedge$ states that for a formula $\bigwedge\Gamma \to (\bigvee\Delta \vee (A \wedge B))$ to be derivable, both formulas $\bigwedge\Gamma \to (\bigvee\Delta \vee A)$ and $\bigwedge\Gamma \to (\bigvee\Delta \vee B)$ have to be derivable.

Figure 2.2 shows a derivation in **G3cp** and its tree structure[4]: as already observed, derivations in sequent calculus have the form of a tree or, more precisely, of a rooted tree.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\nabla}{A \Rightarrow A, B} \quad \dfrac{\overset{\nabla}{A, B \Rightarrow B}}{A \Rightarrow \neg B, B}\,R\neg}{A \Rightarrow A \wedge \neg B, B}\,L\wedge}{\Rightarrow A \wedge \neg B, A \to B}\,R\to}{\neg(A \wedge \neg B) \Rightarrow A \to B}\,\mathsf{L}\neg}{\Rightarrow \neg(A \wedge \neg B) \to (A \to B)}\,\mathsf{R}\to$$

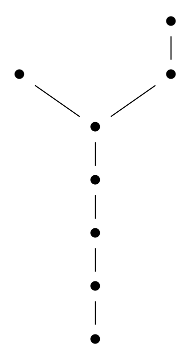


Figure 2.2: Derivation tree

Definition 2.2.2. A *directed graph* G is given by a set of vertices V and a set of edges $E \subseteq V \times V$. For x, y vertices, an edge e is the ordered pair $e = (x, y)$. A *path* p in a graph is a sequence of edge-relating nodes. Formally, a path is a sequence $p = e_1, \ldots, e_n$ such that $(e_i, e_{i+1}) \in E$. A *cycle* is a path in which $e_i = e_k$ for some e_i, e_k element of the path and $i \neq k$.

A *tree* is a directed acyclic graph where there is at most one path relating each pair of nodes. A *rooted tree* is a tree in which one node has been designed to be the root. The *parent* of a node v is the node v' such that $(v', v) \in E$; a child of a node v is a node w such that $(v, w) \in E$. All nodes of the tree have exactly one parent, except for the root, which has no parent. The nodes having no children are called the *leaves* of the

[4]Symbol $\triangledown$ denotes derivability of a sequent: the sequents occurring at the leaves of the derivations are not directly derivable by means of init, since A and B are not necessarily atomic.

tree. A *branch* in the tree is a sequence of nodes which starts from one of the leaves and ends at the root of the tree. The *height* of a finite rooted tree is defined as the number of nodes occurring in its longest branch, minus one.

In what follows, we will refer to *labelled trees*: these are trees in which every node carries some information written in place of the single nodes.

Definition 2.2.3. Derivations in **G3cp** are finite rooted trees labelled with sequents such that:

- The *leaves* of the tree are initial sequents;
- The sequents occupying intermediate nodes in the tree are obtained from the sequent(s) occupying the node(s) directly above them by means of a correct application of an inference rule (both structural or logical);
- The *root* of the tree is the conclusion of the derivation (the *endsequent*).

We denote by $\vdash_{\mathbf{G3cp}}$ derivability of a sequent in **G3cp**: we say a sequent to be derivable if and only if there is a a finite derivation for it. Similarly, a formula A is derivable in **G3cp**, in symbols $\vdash_{\mathbf{G3cp}} A$, if there is a derivation of the sequent $\Rightarrow A$.

Theorem 2.2.1. The sequent calculus **G3cp** is *adequate* with respect to classical propositional logic, i.e., the following hold:

- *Soundness*: $\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta$ implies $\vDash \bigwedge \Gamma \to \bigvee \Delta$;
- *Completeness*: $\vDash \bigwedge \Gamma \to \bigvee \Delta$ implies $\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta$;

where $\vDash A$ signifies that A is a theorem of classical propositional logic.

Proof. The proofs of soundness and completeness for **G3cp** can be found in [102] and [74]. $\square$

We now analyse the structural properties of **G3cp**. Some general definitions are needed.

Definition 2.2.4. Let R be a rule of a sequent calculus **S**. The rule is *admissible* if from derivability of the premiss(es) follows derivability of the conclusion.

Definition 2.2.5. Let R be a rule of **S**. The rule is *invertible* if from derivability of the conclusion follows derivability of the premiss(es).

All the rules of **G3cp** are invertible. The following lemmas assess admissibility of the structural rules of weakening and contraction[5]. Proofs are by induction on the height of a derivation - where the height of a derivation is defined as the height of the corresponding tree (refer to [102] and [74] for the proofs).

Theorem 2.2.2. The structural rules of weakening

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \mathsf{Wk_L} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ \mathsf{Wk_R}$$

are admissible in **G3cp**.

Theorem 2.2.3. The structural rules of contraction

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \mathsf{Ctr_L} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \mathsf{Ctr_R}.$$

[5]Instead of proving admissibility of these rules, we could have included them in the sequent calculus, as originally done by Gentzen and in system **G1** of [102]. A sequent calculus with admissible structural rules, however, is better suited for proof search: refer to [102].

are admissible in **G3cp**.

The structural rule of cut requires some additional discussion. Usually, the rule is necessary to prove completeness of the sequent calculus with respect to the axiom system. This completeness proof consists in showing that the axioms are derivable in the calculus and that, if the premisses of an inference rule are derivable, then its conclusion is derivable (this amounts to prove *admissibility* of the inference rule). More precisely, the cut rule is needed to establish admissibility of the *modus ponens* rule (see Remark 2.2.1). However, occurrences of cut perturb the structure of a derivation, due to the fact that the rule is not analytic.

Definition 2.2.6. A rule is *analytic* if all the formulas occurring in its premisses are subformulas of formulas occurring in the conclusion. A sequent calculus is analytic if all its rules are analytic[6].

All the rules of **G3cp** are analytic except for the cut rule which, read bottom-up, introduces a new formula in the derivation or, read top-down, makes the cut formula disappear from the derivation. The property of analyticity is a crucial requirement in case we do not know whether a formula is derivable or not, and we seek a derivation for it (refer to Section 2.3).

Depending on the way a proof system has been defined, two strategies can be applied to ensure analyticity. If, as in the case of **G3cp**, the cut rule is included in the system, the following theorem has to proved (refer to [102] or [74] for the proof):

Theorem 2.2.4 (Cut elimination)**.** Given a derivation of a sequent $\Gamma \Rightarrow \Delta$ in **G3cp**, it is possible to specify a constructive procedure to transform the **G3cp** derivation into a derivation of the same sequent in which the rule of cut is not used.

The Cut elimination is a normalization theorem, stating that all derivations can be constructively transformed into derivations "in normal form", i.e., where the cut rule is not applied. Otherwise, a system of sequent calculus can be defined *without* adding the cut to the set of rules. In this case, the proof system is analytic; however, to prove completeness of the calculus with respect to the axiom system a proof of cut admissibility is needed. Alternatively, a completeness proof of the sequent calculus with respect to the models for the logic yields as a corollary admissibility of cut, since the set of formulas which are valid in the semantics are proved to be derivable in the cut-free sequent calculus. In the following chapters we shall mainly resort to the definition of such cut-free calculi, for which we either prove - where possible - admissibility of cut, or we show semantic completeness with respect to the class of models. Chapter 4 contains a proof of cut-elimination.

Remark 2.2.1. The inference rule of *modus ponens* (MP) can be regarded as expressing a natural pattern of reasoning, which combines the two proofs of the assumptions into a proof of the conclusion. Since sequent calculus aims at a formal representation of arguments as mathematical demonstrations, it also includes a rule to represent (MP): this is the rule of cut.

$$\frac{\vdash A \quad \vdash A \to B}{\vdash B}\ (\mathrm{MP}) \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

The rule of cut is needed to prove admissibility of (MP) in **G3cp**. Assume that $\vdash A$, i.e., that there is a derivation of $\Rightarrow A$ and $\vdash A \to B$, i.e., that there is a derivation of

[6] The notion of subformula is standardly defined as follows: formula A is a subformula of A, and formulas A and B are subformulas of $A \wedge B$, $A \vee B$ and $A \to B$. However, to have analyticity of some calculi, as the labelled ones, the notion of subformula has to be relaxed, as to include also labels (refer to Chapter 3). More generally, we can say that a rule is analytic if the formulas occurring in the premisses are generated by formulas occurring in the conclusion.

$\Rightarrow A \to B$. By invertibility of rule $\mathsf{R}\to$, we have a derivation of $A \Rightarrow B$. Application of the cut rule yields $\Rightarrow B$, thus $\vdash B$.

$$\frac{\Rightarrow A \quad A \Rightarrow B}{\Rightarrow B}\ \mathsf{cut}$$

Then, thanks to the cut elimination theorem, the occurrence of cut can be eliminated from the derivation of $\Rightarrow B$. Generally speaking, the great advantage of sequent calculus over other proof systems consists in allowing a composition of proofs which is analytical - by means of the cut-elimination theorem.

2.3 Desirable properties of sequent calculus

Given a well-formed formula we are interested in knowing whether this formula is valid in a certain system of logic **L**. If we have a sequent calculus **S** which is sound and complete with respect to **L**, we can decide the validity of the formula by trying to construct a derivation for it. Thus, a system of sequent calculus can be fruitfully employed to provide a decision procedure for the logic.

One of the main interests of a proof theorist is to obtain - where possible - rules which facilitate the definition of an efficient and automated decision procedure. Thus, it is useful to consider derivations *bottom-up*, in what is called *root-first* proof search. The formula to be checked is placed at the root, and one tries to construct a derivation for it applying the inference rules of the system. There are several desirable requirements allowing an efficient root-first proof search. Here follows a non-exhaustive list of properties.

Analyticity. This property - defined in the previous section - is essential for root-first proof search: if the calculus is analytic, each formula occurring in a derivation is a subformula of the formulas occurring lower in the derivation branch. Thus, at each step of the derivation no new formulas are introduced, and all formulas can be ultimately traced down to the root of the derivation tree. All the (analytic) logical rules of **G3cp** are straightforwardly applicable bottom-up: it suffices to "decompose" the main formula of each rule into its active (sub)formulas. Conversely, applying bottom-up the cut rule requires to perform a guess on what formula should be chosen as cut-formula.

Avoidance of backtrack. In root-first proof search, "backtracking" is the action of going back to a certain sequent and apply a different rule to it. In some calculi the order of application of the rules, or the formula to which a certain rule is applied, can determine the derivability of the formula at the root. In such calculi, to check the derivability of a formula we have to check all possible derivation trees - for if we end up with a failed proof search, it might be that we applied the wrong rules, or in the wrong order, to the sequents.
Usually, it is possible to avoid backtracking by means of an attentive definition of the inference rules. If there is no need to backtrack, construction of just *one* derivation tree is enough to decide derivability of a sequent. Sequent calculi composed only of *invertible* rules avoid backtrack. If a rule is invertible, no pieces of information are lost when going from the conclusion to the premiss(es). For this reason, the invertible version of rule $\mathsf{R}\vee$ (on the left) is preferred over its non-invertible version (on the right, composed of a

pair of rules)[7].

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ \mathsf{R}\vee \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}\ \mathsf{R}\vee' \quad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}\ \mathsf{R}\vee''$$

Observe that, if we were to apply root-first rule $\mathsf{R}\vee'$, we would loose formula B, which could have been the formula allowing to obtain an initial sequent. As explained in [70], invertible inference rules preserve both the validity and the refutability of sequents, allowing to construct only *one* derivation tree to asses the derivability of the sequent placed at the root. Conversely, non-invertible inference rules preserve only the validity of the sequent and thus possibly lead to a division of the proof search tree into *multiple* derivation trees, which are all to be tested in order to ascertain the refutability of the endsequent.

Termination. When attempting to construct a derivation bottom-up, it is essential for the procedure to come to an end. The final result will be either a derivation for the formula at the root (a tree in which all the sequents occurring in the leaves are initial sequents), or a failed proof search tree (a tree in which at least one of the sequents occurring the leaves is a sequent to which no rules are applicable). In **G3cp**, sequents to which no rule is applicable are composed only of atomic formulas, and thus proving termination is immediate. However, it might not be so. In particular, in order to obtain invertibility we may need to add the principal formula of a rule into its the premiss(es). This will most likely introduce a *loop* in the derivation tree, since the same rule could be indefinitely applied to the same formula. In this case, the proof search does not terminate.
By means of an example, consider the rule $\mathsf{L}\to^i$ of intuitionistic sequent calculus **G3i** [102]. The rule repeats formula $A \to B$ in the left premiss, and might lead to a loop in root-first proof search, as shown below.

$$\frac{A \to B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C}\ \mathsf{L}\to^i \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{A \to B \Rightarrow B, A, A}}{A \to B \Rightarrow B, A}\ \mathsf{L}\to^i \quad \dfrac{\triangledown}{B \Rightarrow B}}{A \to B \Rightarrow B}\ \mathsf{L}\to^i}{\Rightarrow (A \to B) \to B}\ \mathsf{R}\to$$

In the leftmost branch an instance of an initial sequent is never reached; moreover, rule $\mathsf{L}\to^i$ is always applicable.
In case of looping derivations, termination can achieved by defining a loop-check procedure. This can be done either by specifying some restrictions on the application of the rules in the proof-search algorithm or by adding some provisos on the application of the sequent calculus inference rules.

Efficiency. A decision problem is a problem which has only two possible solutions: yes or no. If we formulate the decision problem in an adequate formal language and measure the resources needed to solve it (usually in terms of time and space) we can classify the problem into a certain complexity class. When considering a logic, the decision problem consists in finding whether a certain formula is valid in the system. Among other methods, logics can be classified in different complexity classes by looking at the "best" complexity result on the worst cases of their decision procedures. Several complexity results are already known; thus, a decision procedure can be classified as *optimal* if it is no more complex than the "best" complexity result known for the system.

[7] The leftmost version of the rule is usually called "multiplicative", and the rightmost version (composed of two rules) "additive". Refer to [102] for a deeper treatment.

A sound system of sequent calculus offers an immediate decision procedure for the corresponding logic: it suffices to try to build bottom-up a derivation tree. Invertibility of the rules ensures that the construction of one derivation tree is sufficient, and termination of the procedure guarantees the decidability of a calculus. The complexity of the decision procedure via root-first proof search has to be evaluated case by case. However, as we shall see, it is not easy to obtain the result of maximal efficiency - an optimal decision procedure.

The requirements listed above can be classified as desirable computational properties, that is to say, property that are to be sought to obtain an efficient root-fist proof search. However, there could be other aims than computational efficiency at play in the definition of the rules of a sequent calculus. For instance, we could be interested in the expressive power of the sequent calculus, or in the fact that rules should define the operational meaning of the connectives. We list some additional properties of wider interest, which will be useful in the chapters to follow[8].

Separation. In Gentzen's original systems, the sequent rules for a connective can be used to give a purely syntactic account of the connective's meaning. For this reason, the rules should not introduce in the conclusion connective others and than the principal one: this is the requirement of *separation*. Furthermore, the rules should be *symmetric* (one left and one right rule).

Locality and context-independance. Both notions concern the rules schemata, which should be general enough to allow the application of the rules immediately, without checking any side conditions or requirements on the context. According to *locality*, the conclusion of a rule should depend only on its premiss(es), and not on other global conditions which might be imposed on derivations. The notion of *context-independence* regards the context of a rule: there should be no restrictions on the formulas occurring in the context and, moreover, the context should not be modified between the conclusion and the premiss(es) of the rule.

Modularity. The rules should be defined in such a way as to modularly capture logics of the same family: calculi for stronger systems should be defined by adding rules to the basic systems.

Standard calculi. A sequent calculus is *standard* if it composed of a finite number of rules, each with a finite number of premisses. It is possible to define *non-standard* systems: calculi with an infinite set of rules or with an unbound number of premisses. While such calculi might be viewed as "bad" if one seeks rules that define the meaning of the main connectives, defining non-standard systems of rules might be a good solution in terms of efficacy of root-first proof search. In this thesis only standard standard sequent systems are presented.

2.4 Sequent calculi for modal logics

Before considering sequent calculi for conditional logics, we briefly analyse sequent calculi for modal logics, taking **S5** as a case study[9].

[8] Refer to [104] for a list of desirable properties of sequent calculi.

[9] For an introduction to modal logic, as well as for an axiomatization for **S5**, refer to Appendix 1.A. For surveys on the proof theory for modal logic, refer to [26] and [105].

The well-formed formulas $\mathcal{F}^{\Box}$ of modal logic **S5** are generated by the following grammar: $p \mid \neg A \mid \bot \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid \Box A$, where $\Box$ is a one-place modal operator.

System of sequent calculi for modal logics can be given by extending **G3cp** with rules for the modal operator. Fitting was the first to provide internal calculi for modal logics, in [25]. Many other proof systems have been devised since then. For instance, a sequent calculus for **K** extends **G3cp** with the following rule[10]:

$$\frac{\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Delta'}\ \mathsf{K}$$

where $\Box\Gamma$ is a sequence of formulas beginning with $\Box$; for instance, in case $\Gamma = B_1, \ldots, B_k$, $\Box\Gamma = \Box B_1, \ldots, \Box B_k$.

Rule K allows to derive the axiom of distributivity $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$. In case Γ, Γ' and Δ' are empty, the rule corresponds to the rule of necessitation: if A is a theorem of **K**, so is $\Box A$. However, the rule lacks the property of separation, since the conclusion displays more than one occurrence of $\Box$. Moreover, the rule is not invertible, and it is not uniquely determined how the rule should be applied bottom-up, since Γ' could contain boxed formulas, which get deleted with application of the rule.

Remark 2.4.1. To interpret the rules of a sequent calculus system it can be useful to resort to the notion of validity of a sequent.

Given a Kripke model $\mathcal{M}$ for modal logics **K**, a sequent $\Gamma \Rightarrow \Delta$ is valid at $\mathcal{M}$, in symbols $\mathcal{M} \vDash \Gamma \Rightarrow \Delta$, if for all the worlds x of the model it holds that if x forces all the formulas in Γ, then it forces at least one of the formulas in Δ. Conversely, a sequent is not valid at $\mathcal{M}$, in symbols $\mathcal{M} \vDash \Gamma \Rightarrow \Delta$, if there is a world x in $\mathcal{M}$ such that x forces all the formulas in Γ and falsifies all formulas in Δ. A sequent is valid, in symbols $\vDash \Gamma \Rightarrow \Delta$, if it is valid at all the models. A sequent is not valid, in symbols $\nvDash \Gamma \Rightarrow \Delta$, if it is not valid in at least one model.

Since we want the rules to be sound, we require they preserve validity: given a rule (R), if its premiss(es) S_1, S_2 are valid, then its conclusion C is valid:

$$\vDash S_1 \textit{ and } \vDash S_2 \textit{ implies } \vDash C$$

If we apply this interpretation to rule K, we obtain the following, which amounts to a proof of soundness of the rule. Suppose the premiss of K, sequent $\Gamma \Rightarrow A$, is valid in all the models. Let us consider an arbitrary model $\mathcal{M}$. Then, for all the worlds y in $\mathcal{M}$ it holds that if y forces all formulas in Γ then y forces A, $y \Vdash A$. It holds that the model $\mathcal{M}$ satisfies the conclusion of K. Let x be an arbitrary world of the model. We want to show that if x forces all the formulas in $\Box\Gamma$ (along with possibly other formulas) then $x \Vdash \Box A$, i.e. that if for all worlds y of the model such that xRy it holds that y forces all formulas in Γ, then it holds that for all worlds y of the model such that xRy, $y \Vdash A$. If the world x does not see any other worlds, the condition is vacuously satisfied, and there are possibly other formulas Γ', Δ' valid at the model. If there are some worlds y such that xRy, the condition is satisfied by validity of the premiss: all worlds in the model forcing all formulas in Γ force A as well.

In the following, it will often be useful to interpret the rules as preserving validity "bottom-up", which amounts to prove the counterpositive statement:

$$\nvDash C \textit{ implies } \nvDash S_1 \textit{ or } \nvDash S_2$$

[10]The rule is a multi-succedent version of the rules from [82], which also presented single-succedent rules for **T** and **S4**. A sequent calculus is single-succedent if only one formula is allowed to occur in the consequent of a sequent.

meaning that, if there is a model which falsifies the conclusion, then there is a model that falsifies at least one of the premisses.
Let us consider rule K. If the conclusion of the rule is not valid, there is a model $\mathcal{M}$ such that $\mathcal{M} \nvDash \Gamma', \Box\Gamma \Rightarrow \Box A, \Delta'$. Thus, there exists a world x in the model such that $x \Vdash G'$ for all $G' \in \Gamma'$, $x \Vdash \Box G$ for all $G \in \Gamma$, $x \nVdash \Box A$ and $x \nVdash F$ for all $F \in \Delta'$. From $x \nVdash \Box A$ we have that there exists $y \in W$ such that xRy and $y \nVdash A$. From $x \Vdash \Box G$ for all $G \in \Gamma$, we obtain $y \Vdash G$, for all G. Thus, from a model which falsifies the conclusion we define a model which falsifies the premiss, i.e., a countermodel of $\Gamma \Rightarrow A$.
Most notably, according to this interpretation, the formulas occurring in the antecedent of sequents can be considered as true, and the formulas occurring in the consequent can be considered as false[11].

Calculi for extensions of modal logic **K** can be defined by adding to the calculus for **K** rules in correspondence with the semantic properties of the logics. For instance, a sequent calculus for **S4** results from the addition of the following rules, which are a weakened version of the rules from [101].

$$\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ \mathsf{T} \qquad\qquad \frac{\Box\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Delta'}\ 4$$

Rule T is local, context-independent and has the property of separation, while rule 4 is local, but context-dependent and lacking the property of separation, since the formulas occurring in the antecedent needs to be boxed.

Finally, a calculus for **S5** can be defined by adding to **G3cp** rules K, T and the following rule, corresponding to the semantic condition of Euclideaness [101]:

$$\frac{\Box\Gamma \Rightarrow A, \Box\Delta}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Box\Delta, \Delta'}\ 45$$

The rule is context-dependent, but it has a worse disadvantage: it is not possible to prove cut elimination for a sequent calculus with 45. To derive axiom (B) $A \to \Box\Diamond A$ we need to use both 45 and the cut rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{\triangledown}{\Box\neg A \Rightarrow \Box\neg A}}{\Rightarrow \neg\Box\neg A, \Box\neg A}\ \mathsf{L}\neg}{\Rightarrow \Box\neg\Box\neg A, \Box\neg A}\ 45 \qquad \dfrac{\dfrac{\dfrac{\triangledown}{A \Rightarrow A}}{\neg A, A \Rightarrow}\ \mathsf{L}\neg}{\Box\neg A, A \Rightarrow}\ \mathsf{T}}{\dfrac{A \Rightarrow \Box\neg\Box\neg A}{\Rightarrow A \to \Box\neg\Box\neg A}\ \mathsf{R}\to}\ \mathsf{cut}$$

Thus, to prove completeness of the sequent calculus with respect to the axiom system we need cut; but the occurrence of the rule cannot be eliminated from the derivation.

Remark 2.4.2. The case of the cut-elimination proof that fails is the one in which the left and right premisses of cut are derived by 45 and T respectively, and in which the cut formula is $\Box A$ and is not principal in 45:

$$\dfrac{\dfrac{\vdots}{\dfrac{\Box\Gamma, \Sigma \Rightarrow \Box\Delta', D, \Pi, \Box A}{\Box\Gamma, \Sigma \Rightarrow \Box\Delta', \Box D, \Pi, \Box A}}\ 45 \qquad \dfrac{\vdots}{\dfrac{A, \Sigma \Rightarrow \Pi}{\Box A, \Sigma \Rightarrow \Pi}}\ \mathsf{T}}{\Box\Gamma, \Sigma \Rightarrow \Box\Delta', \Box D, \Pi}\ \mathsf{cut}$$

The cut-elimination proof proceeds by induction, usually showing that each occurrence of cut can be permuted upwards in the derivation, or that the cut can be applied to

[11] This interpretation corresponds to the construction of a refutation tree for the formula, as done with tableaux calculi.

formulas of a smaller complexity than the original cut formula. The base case of the induction then shows how the cut can be eliminated from the initial sequents. In the case described above, however, we are stuck: we cannot apply the cut to formulas of smaller complexity, and if we permute the rule to formulas occurring higher in the derivation, we do not obtain the conclusion of cut (formula D is missing a $\Box$):

$$\dfrac{\dfrac{\vdots}{\Box\Gamma, \Sigma \Rightarrow \Box\Delta', D, \Pi, \Box A} \quad \dfrac{\dfrac{\vdots}{A, \Sigma \Rightarrow \Pi}}{\Box A, \Sigma \Rightarrow \Pi}\,\mathsf{T}}{\Box\Gamma, \Sigma \Rightarrow \Box\Delta', D, \Pi}\,\mathsf{cut}$$

Thus, the proof of completeness of the calculus with respect to the axiomatization fails. Up to now, a Gentzen-style and cut-free sequent calculus for **S5** does not exist[12].

To obtain cut-free, invertible and local proof systems for modal logic, and in particular for **S5**, the solution is to enrich the formalism of sequent calculus defined for classical propositional logic. This can be done in two ways: either by enriching the *language* of the calculus, or by enriching the *structure* of sequents. The first approach might be qualified as *semantic*, since *labels*, i.e., variables representing the semantic elements of the models, are added to the language. We add to the language formulas as $x : A$, expressing validity of formula A at x, label representing a world of the Kripke model. The labelled approach has also been qualified as *external*, meaning that some elements "external" to the syntax are introduced into the calculus. However, the definition of such "external" elements might be vague. Most notably, a labelled sequent calculus does not have a direct formula interpretation of sequents in the language of the logic.

The second approach consists in allowing extra structural connectives in addition to the sequent arrow "$\Rightarrow$" and the commas "$,$". For instance, the hypersequent calculi introduce the structural connective "$|$", and nested sequents use the structure "$[\ ]$". In these calculi it is possible to define a direct formula interpretation for sequents, as done for **G3cp**; for this reason, these calculi are usually referred to as *internal*, or *pure*.

In the course of next chapters we present both labelled and internal proof systems for conditional logics. However, before that, it is useful to look at internal and labelled proof systems defined for modal logics.

2.5 A semantic approach

The first sequent calculus internalizing semantic informations is due to Kanger, in [44]. He developed a proof system for modal logic **S5** employing *spotted* formulas, i.e., formulas indexed with "marks", the natural numbers. However, Kanger did not assign any meaning to its marks, considering them as syntactic elements external to the language. After Kanger's seminal work, many other contributions (among which Kripke in [49], Fitting in [25], Gabbay in [29] and Viganò in [103]) devised formal systems, not necessarily of sequent calculus, which internalized the semantics of a logic into a proof system, by means of prefixed formulas, labelled formulas or semantic tableaux.

We briefly present labelled sequent calculi for modal logic. Negri in [69] introduced a family of labelled systems able to modularly capture a wide number of modal logics[13]. Labelled systems for modal logics introduce in the language $\mathcal{F}^{\Box}$ of modal logics new

[12] More general limitative results on Gentzen proof systems for modal logic can be found in [54]. This does not mean that it is not possible to define cut-free proof systems for **S5**: as we shall see, such calculi can be defined by enriching the formalism of sequent calculus.

[13] More precisely, the calculi are apt to capture any modal logic characterized by first-order conditions on their Kripke frames [23].

elements, called *labels*, which are symbols $x, y, z...$ denoting the worlds of a Kripke frame. Let $\mathbf{K}^*$ denote one of the following modal logics: **K**,**T**, **S4**, **S5**. The labelled calculi for $\mathbf{K}^*$ are called $\mathbf{G3K}^*$.

Definition 2.5.1. A labelled sequent is an expression of the form

$$\Gamma \Rightarrow \Delta$$

where Γ, Δ are multisets of of expressions $x : A$ (denoting the forcing relation $x \Vdash A$) for x ranging in an infinite set $W = \{x, y, z...\}$, whose elements we call *labels*, and A formula of the language of modal logic **K***. Γ may as well contain expressions of the form xRy, called *relational atoms*, which stand for the accessibility relation between worlds.

Initial sequents

$$\dfrac{}{x : p, \Gamma \Rightarrow \Delta, x : p}\ \mathsf{init} \qquad \dfrac{}{x : \bot, \Gamma \Rightarrow \Delta}\ \bot_{\mathsf{L}}$$

Propositional and modal rules

$$\dfrac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge \qquad \dfrac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B}\ \mathsf{R}\wedge$$

$$\dfrac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \dfrac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\ \mathsf{R}\vee$$

$$\dfrac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \to B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\to \qquad \dfrac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \to B}\ \mathsf{R}\to$$

$$\dfrac{\Gamma \Rightarrow \Delta, x : A}{x : \neg A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\neg \qquad \dfrac{x : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : \neg A}\ \mathsf{R}\neg$$

$$\dfrac{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta}{x : \Box A, xRy, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Box \qquad \dfrac{xRy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\ \mathsf{R}\Box\ (y!)$$

Rules for semantic conditions

$$\dfrac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{Rf} \qquad \dfrac{xRz, xRy, yRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}\ \mathsf{Tr}$$

$$\dfrac{yRx, xRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\ \mathsf{Sym}$$

Condition $(y!)$ means that label y does not occur in the conclusion.
G3K = Initial sequents + propositional and modal rules;
G3T= **G3K**+ Rf; **G3S4** = **G3T** + Tr; **G3S5** = **G3S4**+ Sym.

Figure 2.3: Sequent calculi $\mathbf{G3K}^*$

The rules of the calculi, given in Figure 2.3, are defined by analysing the truth conditions of formulas in a Kripke model. Recall that the truth condition for a boxed formula is the following:

$$x \Vdash \Box A \text{ iff for all } y \text{ such that } xRy \text{ it holds that } y \Vdash A.$$

The rules can be interpreted keeping in mind that refutability needs to be ensured in bottom-up proof search, as explained in Remark 2.4.1. Thus, rule $\mathsf{L}\Box$ details the condition for a boxed formula true at a world x: in any world seen from x, A is true, and this reasoning can be applied to all the worlds xRy present in the sequent. Conversely, rule R$\Box$ illustrates the semantic condition for a false boxed formula: there exists a world (a new world, in which no hypothesis are made) such that xRy and in which A is not true.

Example 2.5.1. Derivation of axiom (B) $A \to \Box\Diamond A$ in calculus **G3S5**.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\bigtriangledown}{xRy, yRx, x : A, y : \Box\neg A \Rightarrow x : A}}{xRy, yRx, x : A, y : \Box\neg A, x : \neg A \Rightarrow}\, \mathsf{L}\neg}{xRy, yRx, x : A, y : \Box\neg A \Rightarrow}\, \mathsf{L}\Box}{xRy, x : A, y : \Box\neg A \Rightarrow}\, \mathsf{Sym}}{xRy, x : A \Rightarrow y : \neg\Box\neg A}\, \mathsf{R}\neg}{x : A \Rightarrow x : \Box\neg\Box\neg A}\, \mathsf{R}\Box}{\Rightarrow x : A \to \Box\neg\Box\neg A}\, \mathsf{R}\to
$$

The rules of **G3K*** display a modular structure: Rules for semantic conditions are defined in correspondence with the frame properties of a Kripke model for the corresponding logic. Thus, a sequent calculus for modal logic **K** is composed only of propositional rules and of the rules for $\Box$; while sequent calculus for **T** displays one additional rule, corresponding to reflexivity of the frames. Moreover, all the rules respect the separation property. Some rules have some side conditions (R$\Box$ has the proviso that y should be a new label), and thus the sequent calculi cannot be classified as fully local, since the side condition on the context has to be verified for the rule to be applicable. Strictly speaking, the rules do not display the subformula property, since there are some formulas (relational atoms) which are not subformulas of any formula occurring in the derivation. This is what happens also with sequent calculi for first order logic; a more relaxed notion of subformula, discussed in Chapter 3, solves the problem.

Labelled sequent calculi **G3K*** enjoy strong structural properties: all the rules are invertible, and the rules of weakening and contraction are height-preserving admissible. The rule of cut can be proved to be admissible in the calculi.

Remark 2.5.1. In order to obtain admissibility of contraction for a labelled calculus, contracted instances of the rules might need to be added. Such rules can be defined applying the so-called *closure condition* [74]. According to this condition, if it is the case that a rule might display two instances of the principal or active formulas, to the calculus it is added a rule in which the duplicated formulas are contracted into one.
In the case of **G3K***, one of the rules added by the closure condition would be the following, generated by Tr in the case $x = z$:

$$
\cfrac{xRx, xRy, yRx, \Gamma \Rightarrow \Delta}{xRy, yRx, \Gamma \Rightarrow \Delta}\, \mathsf{Tr}^*
$$

However, this rule does not need to be added, since its premiss can be derived by application of rule Ref.

The rules are sound and complete with respect to the Kripke models for the corresponding logic [69]. Since the labelled sequents do not have a direct formula interpretation in the language of **K***, the proof of soundness requires the definition of a *realization*, i.e., an interpretation the new elements of the language (the labels and the relational atoms) within Kripke models. Due to their strong link with the semantics, proofs of completeness for calculi **G3K*** proceed in a quite straightforward way, by showing how a countermodel can be constructed from the formulas occurring in a failed derivation branch. Conversely, in order to establish termination we need to define a proof-search strategy, preventing the loops which otherwise would occur in root-first proof search.

2.6 A syntactic approach

The so-called internal calculi enrich the structure of sequents in various ways. In this section are presented two kinds of internal calculi, in order of increasing generality: hypersequents and nested sequents. Hypersequents provide a cut-free proof system for **S5**; the formalism of nested sequents will be useful in the treatment of conditional logics.

Hypersequents

Hypersequents can be thought of as a generalization of Gentzen's sequent calculi: in this formalism, several sequents are considered in parallel, and might interact with each other. Hypersequents introduce a new structural connective, the vertical bar, which represents disjunction at the meta-level.

The idea behind hypersequents goes back to Kripke, who proposed in [48] semantic tableaux for **S5**. Mints presented a tableaux system in the form of a Gentzen sequent calculus, in which formulas were labelled with worlds [65, 66], implicitly employing the hypersequent method [9]. The first to present hypersequents was Pottinger, in a half-page-long abstract in which he gave rules for systems **T**, **S4** and **S5**, but no proof [91]. Avron developed a system of hypersequents for **S4** and **S5** and applied the hypersequent method to other non-classical systems [9].

Hypersequents for modal logic were studied by Restall in [94], Poggiolesi in [87] and Kurokawa in [50]. Restall's version of hypersequents for **S5** is the simpler to prove cut elimination, and we here present a slight variation of his system[14].

Definition 2.6.1. A hypersequent is an object of the form:

$$\mathcal{G} \ = \ \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

in which each $\Gamma_i \Rightarrow \Delta_i$ is called a "component" of the hypersequent. The intended interpretation of a component of the hypersequent is the usual one: $(\Gamma_i \Rightarrow \Delta_i)^{int} := \bigwedge \Gamma_i \rightarrow \bigvee \Delta_i$. The formula interpretation of a hypersequent in the language of **S5** is the following:

$$(\mathcal{G})^{int} := \Box(\Gamma_1 \Rightarrow \Delta_1)^{int} \vee \Box(\Gamma_2 \Rightarrow \Delta_2)^{int} \vee \cdots \vee \Box(\Gamma_n \Rightarrow \Delta_n)^{int}$$

We can read each component of the hypersequent as keeping track of the formulas relative to a world of a Kripke model for **S5**. Thus, rule $\mathsf{R}\Box$ can be read bottom-up as follows (refer to Remark 2.4.1): for a formula $\Box A$ to be false at some world x, it means that there exists a world y seen by x in which A is false. Thus, the rule creates a new component (a new world) and stores the information that A is false. Similarly, rule $\mathsf{L}\Box$ is saying that if a formula $\Box A$ true at some world x, then the formula A must true at all the worlds seen by x; thus, A can be copied in the antecedent of any component of the hypersequent, representing the words that are seen from the actual world. However, since we are considering **S5**, there is no need to specify which the actual world is, as the accessibility relation is universal. Rule $\mathsf{L}\Box$ is a transfer rule, allowing to formulas to "pass" between different components. Interaction between components is the key feature of hypersequent calculus[15].

[14] For a comprehensive presentation of the various hypersequent systems for **S5** and a comparative analysis refer to [15]. Other fruitful applications of the hypersequent method are on the proof theory of the Gödel-Dummett intermediate logic [20, 10] and on the proof theory of fuzzy logics [64].

[15] The most interesting rule showing the interaction between sequents in hypersequents calculi is possibly rule of com, which is needed to give a hypersequent calculus for Gödel-Dummet logic:

$$\frac{\mathcal{G}|\Gamma, \Delta \Rightarrow A \quad \mathcal{G}' \mid \Gamma', \Delta' \Rightarrow B}{\mathcal{G}|\mathcal{G}'|\Gamma, \Gamma' \Rightarrow A|\Delta, \Delta' \Rightarrow B} \ \mathsf{com}.$$

Initial sequents

$$\frac{}{\mathcal{G} \mid p, \Gamma \Rightarrow \Delta, p}\ \mathsf{init} \qquad \frac{}{\mathcal{G} \mid \bot, \Gamma \Rightarrow \Delta}\ \bot_{\mathsf{L}}$$

Propositional and modal rules

$$\frac{\mathcal{G} \mid A, B, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid A \wedge B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \wedge B}\ \mathsf{R}\wedge$$

$$\frac{\mathcal{G} \mid A, \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid B, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid A \vee B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A, B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \vee B}\ \mathsf{R}\vee$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \quad \mathcal{G} \mid B, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid A \to B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\to \qquad \frac{\mathcal{G} \mid A, \Gamma \Rightarrow \Delta, B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \to B}\ \mathsf{R}\to$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A}{\mathcal{G} \mid \neg A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\neg \qquad \frac{\mathcal{G} \mid A, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \neg A}\ \mathsf{R}\neg$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Box A}\ \mathsf{R}\Box$$

$$\frac{\mathcal{G} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Box A, \Sigma \Rightarrow \Pi}{\mathcal{G} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}\ \mathsf{L}\Box \qquad \frac{\mathcal{G} \mid \Box A, A, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Box A, \Gamma \Rightarrow \Delta}\ \mathsf{T}$$

Figure 2.4: Sequent calculus $\mathcal{H}_{\mathbf{S5}}$

Example 2.6.1. Derivation of axiom (B) $A \to \Box\neg\Box\neg A$:

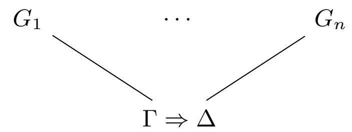
$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\bigtriangledown}{A, \Box\neg A, A \Rightarrow A | \Box\neg A \Rightarrow}}{\neg A, \Box\neg A, A \Rightarrow | \Box\neg A \Rightarrow}\ \mathsf{L}\neg}{\Box\neg A, A \Rightarrow | \Box\neg A \Rightarrow}\ \mathsf{T}}{A \Rightarrow | \Box\neg A \Rightarrow}\ \mathsf{L}\Box}{A \Rightarrow | \Rightarrow \neg\Box\neg A}\ \mathsf{R}\neg}{A \Rightarrow \Box\neg\Box\neg A}\ \mathsf{R}\Box$$

The rules of $\mathcal{H}_{\mathbf{S5}}$ are sound (i.e., they preserve validity) under the formula interpretation given before. A proof of cut admissibility for the calculus can be found in [94]. Moreover, the rules of $\mathcal{H}_{\mathbf{S5}}$ are local and respect the separation property. Modularity, however, cannot be fully achieved, as hypersequents are an ideal structure to represent "parallel" elements, which are equivalent to one another, as the worlds of a Kripke model for **S5**. Thus, to capture weaker accessibility relations we need the more refined structure of nested sequents.

Nested sequents

Nested sequents were first introduced by Kashima in [45], and independently defined by Brünnler, with the name "deep sequents" [17], and by Poggiolesi, with the name of "tree-hypersequents" [88]. Nested sequents are a generalization both of Gentzen sequents, which are nested sequents of depth zero, and of hypersequents, which are nested sequents of depth one [26].

Nested sequent are trees of sequents, in which rules can be applied "deep" up in the tree, and not only to the formula occurring at the root sequent. Intuitively, to a nested sequent $\Gamma \Rightarrow \Delta, [G_1], \ldots, [G_n]$ where $G_1, \ldots G_n$ are nested sequents, there corresponds the following tree:



Brünnler presents a family of nested system, which modularly capture a family of normal modal logics, including **S5** [17]. His aim is to "develop proof systems with the same good properties of Negri's labelled systems but do so without using labels" [17]. Nested sequents are modular as labelled sequent calculi, but stay within the language and allow for a true subformula property as hypersequents. As a drawback, however, they are far from optimal: they can be of exponential size in root-first proof search.

We here present a two-sided nested calculus for **S5**, obtained from the one-sided version given in [17]. Refer to [17] for a presentation of the fully modular system for modal logics.

Remark 2.6.1. We defined two-sided sequents, $\Gamma \Rightarrow \Delta$, with formula interpretation $\bigwedge \Gamma \rightarrow \bigvee \Delta$. One-sided sequent are sequents $\Rightarrow \Delta$, with formula interpretation $\bigvee \Delta$. One- and two- sided sequents are equivalent: $\Gamma \Rightarrow \Delta \;\; = \;\Rightarrow \overline{\Gamma}, \Delta$, where for $\Gamma = A_1, \ldots, A_n$, $\overline{\Gamma} = \neg A_1, \ldots, \neg A_n$. A one-sided sequent calculus displays one-sided sequents.

Definition 2.6.2. Two-sided nested sequents have the following form:

$$\Gamma \Rightarrow \Delta, [G_1], \ldots, [G_n]$$

where each G_i is in turn a two-sided nested sequent. The interpretation is given by the following formula:

$$(\Gamma \Rightarrow \Delta, [G_1], \ldots, [G_n])^{int} := \bigwedge \Gamma \rightarrow \bigvee \Delta \vee \Box(G_1)^{int} \vee \cdots \vee \Box(G_n)^{int}$$

To operate with nested sequents we need to define the notion of context, denoting a unique "hole" in the nested sequent. In the two-sided nested calculus, the hole will be filled with a whole sequent. Rules of $\mathcal{N}_{\mathbf{S5}}$ are given in Table 2.5.

Definition 2.6.3. We define a context $G\{\ \}$ as:

- $G\{\ \} = \Gamma \Rightarrow \Delta, \{\ \}$ is a context;
- if $F\{\ \}$ is a context, then $G\{\ \} = \Gamma \Rightarrow \Delta, [F\{\ \}]$ is a context.

The nested sequent obtained by filling a context $G\{\ \}$ with a nested sequent $\Gamma \Rightarrow \Delta$, denoted as $G\{\Gamma \Rightarrow \Delta\}$, is defined as:

- If $G\{\ \} = \Sigma \Rightarrow \Pi, \{\ \}$, then $G\{\Gamma \Rightarrow \Delta\} = \Gamma, \Sigma \Rightarrow \Pi, \Delta$;
- If $G\{\ \} = \Sigma \Rightarrow \Pi, [F\{\ \}]$ then $G\{\Gamma \Rightarrow \Delta\} = \Sigma \Rightarrow \Pi, [F\{\Gamma \Rightarrow \Delta\}]$.

The structure of nested sequents mirrors the tree frame structure of Kripke models for modal logic. Thus, each sequent occurring in the nested sequent can be interpreted as expressing the formulas relative to worlds of a Kripke model: formulas occurring at the root of the sequent tree (unboxed) are formulas relative to the actual world, while formulas occurring in the boxed sequents are formulas relative to worlds seen from the actual world. If we assign to the edges of the tree-sequent the natural orientation going away from the root, we obtain a precise correspondence with the worlds of a Kripke frame. For instance, to the nested sequent

$$A \Rightarrow B, [C \Rightarrow D], [E \Rightarrow F, [G \Rightarrow H]]$$

there correspond the following tree structure and Kripke frame:

Initial sequents

$$\dfrac{}{G\{P, \Gamma \Rightarrow \Delta, P\}}\ \mathsf{init} \qquad\qquad \dfrac{}{G\{\bot, \Gamma \Rightarrow \Delta\}}\ \bot$$

Propositional and modal rules

$$\dfrac{G\{A, B, \Gamma \Rightarrow \Delta\}}{G\{A \wedge B, \Gamma \Rightarrow \Delta\}}\ \mathsf{L}\wedge \qquad\qquad \dfrac{G\{\Gamma \Rightarrow \Delta, A\} \quad G\{\Gamma \Rightarrow \Delta, B\}}{G\{\Gamma \Rightarrow \Delta, A \wedge B\}}\ \mathsf{R}\wedge$$

$$\dfrac{G\{A, \Gamma \Rightarrow \Delta\} \quad G\{B, \Gamma \Rightarrow \Delta\}}{G\{A \vee B, \Gamma \Rightarrow \Delta\}}\ \mathsf{L}\vee \qquad\qquad \dfrac{G\{\Gamma \Rightarrow \Delta, A, B\}}{G\{\Gamma \Rightarrow \Delta, A \vee B\}}\ \mathsf{R}\vee$$

$$\dfrac{G\{\Gamma \Rightarrow \Delta, A\} \quad G\{B, \Gamma \Rightarrow \Delta\}}{G\{A \rightarrow B, \Gamma \Rightarrow \Delta\}}\ \mathsf{L}\rightarrow \qquad\qquad \dfrac{G\{A, \Gamma \Rightarrow \Delta, B\}}{G\{\Gamma \Rightarrow \Delta, A \rightarrow B\}}\ \mathsf{R}\rightarrow$$

$$\dfrac{G\{\Gamma \Rightarrow \Delta, A\}}{G\{\neg A, \Gamma \Rightarrow \Delta\}}\ \mathsf{L}\neg \qquad\qquad \dfrac{G\{A, \Gamma \Rightarrow \Delta, \}}{G\{\Gamma \Rightarrow \Delta, \neg A\}}\ \mathsf{R}\neg$$

$$\dfrac{G\{\Gamma \Rightarrow \Delta, [\Rightarrow A]\}}{G\{\Gamma \Rightarrow \Delta, \Box A\}}\ \mathsf{R}\Box \qquad\qquad \dfrac{G\{\Gamma, \Box A \Rightarrow \Delta, [A, \Sigma \Rightarrow \Pi]\}}{G\{\Gamma, \Box A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]\}}\ \mathsf{L}\Box$$

$$\dfrac{G\{A, \Box A, \Gamma \Rightarrow \Delta\}}{G\{\Box A, \Gamma \Rightarrow \Delta\}}\ \mathsf{T} \qquad\qquad \dfrac{G\{\Box A, \Gamma \Rightarrow \Delta, [\Sigma, \Box A \Rightarrow \Pi]\}}{G\{\Gamma \Rightarrow \Delta, [\Sigma, \Box A \Rightarrow \Pi]\}}\ 5$$

Figure 2.5: Nested sequent calculus $\mathcal{N}_{\mathbf{S5}}$

$$\begin{tikzcd}
& G \Rightarrow H \\
C \Rightarrow D & E \Rightarrow F \arrow[u] \\
& A \Rightarrow B \arrow[ul] \arrow[u]
\end{tikzcd} \qquad\qquad \begin{tikzcd}
& w \\
y & z \arrow[u] \\
& x \arrow[ul] \arrow[u]
\end{tikzcd}$$

With this in mind, rules $\mathsf{R}\Box$, $\mathsf{L}\Box$ and T of Table 2.5 can be interpreted in the same way as in the hypersequent calculus $\mathcal{H}_{\mathbf{S5}}$. Rule 5 is needed to characterize **S5** models, with their equivalence relation of accessibility: thus, a boxed formula in some world seen from the actual world can be transferred to a the actual world.

Example 2.6.2. Derivation of axiom (B) $A \rightarrow \Box\neg\Box\neg A$: in $\mathcal{N}_{\mathbf{S5}}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\triangledown}{A \Rightarrow A, [\Box\neg A \Rightarrow\ ]}}{\neg A, A \Rightarrow [\Box\neg A \Rightarrow\ ]}\ \mathsf{L}\neg}{A \Rightarrow [\Box\neg A \Rightarrow\ ]}\ 5}{A \Rightarrow [\ \Rightarrow \neg\Box\neg A]}\ \mathsf{R}\neg}{A \Rightarrow \Box\neg\Box\neg A}\ \mathsf{R}\Box}{\Rightarrow A \rightarrow \Box\neg\Box\neg A}\ \mathsf{R}\rightarrow$$

The results of soundness and completeness for $\mathcal{N}_{\mathbf{S5}}$ can be found in [17]. The structure of nested sequents is richer than the structure of hypersequent: nested sequents allow to distinguish between the actual world (the non-nested sequent) and the worlds “seen” from the actual world, they are apt to capture in a modular way modal logics weaker than **S5**. Refer to [17] for the full family of nested calculi for modal logics.

2.7 To conclude

In the previous sections we presented two classes of proof systems extending the format of Gentzen's sequent calculus: the labelled formalism, enriching the language, and the internal calculi, extending the structure of sequents. Both kinds of calculi display good proof-theoretical properties and are sound and complete with respect to the logics. Both systems are able to capture the family of modal logic in a modular way, and with local and symmetric rules. Which system should be preferred?

The distinction between internal and labelled sequent calculi is somewhat vague, since it relies on the distinction between "internal", or syntactic, and "external", or semantic, elements of the language. It is easy to see how a different definition of the language would change the notion of elements external to it; and moreover, semantic labels could be considered as purely syntactic components - thus forgetting the semantic information they carry.

To appreciate the difference between the labelled and the internal approach, it is useful to consider the distinction between semantically oriented and proof-theoretically proof systems [15]. Semantically oriented calculi are, for instance, semantic tableaux. These systems maintain a strong link with the models of a logic: their main interest consists in assessing the validity or refutability of a formula by attempting to construct a countermodel for the formula. Since the link with the semantics is explicit, the proof of completeness is usually quite immediate; moreover, the rules of these calculi are usually easier to define when compared to the rules of a proof-theoretically oriented calculus. These latter focus instead on efficiency and analyticity, and possibly on the definition of rules which enhance a decision procedure of optimal complexity. Gentzen's original sequent calculus can be classified as a proof-theoretically oriented system: more than on efficiency, he was concerned with the definition of an analytical system whose rules would define the meaning of the operators. The rules of proof-theoretically oriented calculi are analytical, usually local and with the separation property, and they avoid backtrack. Derivations in such calculi are usually shorter than with semantically oriented systems, and the proof of termination might be easier. As a drawback, proving completeness of these calculi with respect to the semantics can be quite difficult. Moreover, the rules of these proof systems can be difficult to find, and possibly less immediate to understand.

The internal calculi for modal logics we presented seem to belong to the class of proof-theoretically oriented calculi: they display short proof-search trees, and termination is relatively easy to prove. On the other hand, establishing completeness with respect to the models for these calculi might be a challenge. In any case, it is still possible to individuate the semantic intuition behind the rules of internal calculi employing, for instance, the interpretation described in Remark 2.4.1.

As for labelled calculi, they can be placed somewhere in between the semantically and proof-theoretically oriented calculi. The rules are defined following the semantic intuition (compare, for instance, rule $\mathsf{R}\Box$ of **G3K**, Figure 2.3, with the internal rule K form Section 2.4). Proof of completeness in these systems is quite straightforward, and the method consists in defining countermodels from failed proof-tree branches. However, these calculi display strong structural properties, such as cut admissibility and invertibility, which place labelled systems close to proof-theoretically oriented calculi.

Thus, on one side, we can find a semantic (model-theoretic) intuition behind internal calculi; on the other, labelled sequent calculi display the strong structural properties of internal systems. Moreover, it is possible to define internal calculi displaying syntactic labels, as Fitting's indexed tableaux [27]; and several results have shown natural translations between labelled and internal systems[16]. Thus, there is no system to be preferred in general: the choice between a labelled and an internal system should depend on the

[16]Refer to [89, pp. 116, 206], [40] and [27].

purpose at hand. For instance, in case a logic has no known system of sequent calculus, the labelled approach is likely to find such an analytic proof system, possibly displaying cut admissibility. Conversely, if one is interested in the definition of automated theorem provers, an internal proof system - if it is found - could be the best option.

2.8 State of the art

In next chapters we present labelled and internal proof systems for families of conditional logics. Several proof systems for conditional logics, both labelled and internal, can be found in the literature. We present here an overview of the main results.

Tableaux systems for $\mathbb{CK}$ were given by Priest in [92]. As for labelled sequent calculi, systems for $\mathbb{CK}$ and extensions were defined by Olivetti, Pozzato and Schwind in [81]. These calculi internalise the set-selection function semantics for the logics by means of two relations, and enjoy cut-elimination. Moreover, a theorem prover has been defined based on their rules. In [33] a tableaux calculus for $\mathbb{PCL}$ and its extensions is presented. Negri and Sbardolini proposed a labelled sequent system for $\mathbb{V}$ based on preferential models with strong connectedness [76], and Negri and Olivetti presented a sequent calculus for $\mathbb{PCL}$ based on neighbourhood models [75]. This last work will particularly relevant for our purposes: next chapter is devoted to define extensions of the labelled calculus to cover all extensions of $\mathbb{PCL}$.

As for internal systems, the first systems defined for conditional logics is from De Swart, who introduced non-standard calculi, displaying an infinite number of rules, or rules with an infinite number of premisses [58]. More recently, Pattinson and Schröder defined in [85] non-standard calculi for $\mathbb{CK}$ and extensions, and Lellmann and Pattinson proposed in [53] non-standard calculi for $\mathbb{V}$ and extensions.

A standard internal calculus for $\mathbb{CK}$ was given by Poggiolesi in [90]. Alenda, Olivetti and Pozzato in [6] gave standard sequent calculi for $\mathbb{CK}$ and extensions by means of nested sequents. Up to now, the only internal and standard sequent calculus defined for $\mathbb{PCL}$ or its extensions (excluding Lewis' logics) is the one given by Nalon and Pattinson in [68]. They define a resolution calculus for $\mathbb{PCL}$; however, their proof system is based on a complex (non-analytic) pre-processing, to transform formulas of the language into formulas of a specific normal form. For our purposes, particularly relevant is [79], in which Olivetti and Pozzato gave a nested sequent calculus adapted for $\mathbb{V}$. Chapter 4 extends the result to logics stronger than $\mathbb{V}$.

Chapter 3

Labelled sequent calculi for $\mathbb{PCL}$ and extensions

At non-normal worlds everything is possible,
and nothing is necessary.
– Graham Priest,
An introduction to non classical logic

In this chapter we present a family of labelled sequent calculi, **G3CL***, for the conditional logic $\mathbb{PCL}$ and its extensions with Lewis' axioms. The sequent calculi are based on neighbourhood semantics; thus, we start by presenting neighbourhood models for these logics, and prove the equivalence of these models with respect to preferential models, the usual class of models for $\mathbb{PCL}$. Then, we give a modular presentation of the systems of rules and their structural properties, along with a proof of cut admissibility for all the systems. After the modular presentation, we introduce two versions of the calculi better suited to capture two sub-families of conditional logics: **G3V***, for the systems featuring nesting and $\mathbf{G3A^*_w}$, for the systems featuring absoluteness. We prove termination and semantic completeness for these calculi.

3.1 Preferential conditional logic

The preferential conditional logic $\mathbb{PCL}$ was first studied by Burgess in [18] and Halpern and Friedman in [28]. This logic is weaker than Lewis' systems, and it can be adapted to capture non-monotonic conditionals: as explained in Chapter 1, the non-monotonic logic **P** introduced by Kraus, Lehman and Magidor corresponds to the flat fragment of $\mathbb{PCL}$.

Definition 3.1.1. The set of well formed formulas of $\mathbb{PCL}$ and its extensions is defined by means of the following grammar, for $p \in Atm$ propositional variable and $A, B \in \mathcal{F}^>$:

$$\mathcal{F}^> ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \to B \mid A > B.$$

The conditional operator $>$ can be read as expressing either a non-monotonic conditional, a counterfactual or a deontic conditional, depending on the system under consideration. When treating logics of Lewis' family we will add to the language the comparative plausibility operator, $\preccurlyeq$, where for $A, B \in \mathcal{F}^>$ the well-formed formula $A \preccurlyeq B$ reads "A is at least as plausible as B".

The semantics of preferential conditional logics is usually defined in terms of preferential models, introduced in Definition 1.4.3 of Chapter 1.

Definition 3.1.2. Extensions of preferential models are defined by specifying the following conditions [28]:

- n *Normality*: For all $x \in W$, W_x is non-empty;
- t *Total reflexivity*: For all $x \in W$ it holds that $x \in W_x$;
- w *Weak centering*: For all $x \in W$, for all $y \in W_x$, it holds that $x \preceq_x y$;
- c *Centering*: For all $x \in W$, for all $y \in W_x$, it holds that $x \preceq_x y$ and if there is $w \in W_x$ such that for all $y \in W_x$, $w \preceq_x y$, then $w = x$;
- u *Local uniformity*: For all $x \in W$, for all $y \in W_x$ it holds that $W_y = W_x$;
- a *Local absoluteness*: For all $x \in W$, for all $y \in W_x$ it holds that $W_y = W_x$ and for all $w_1, w_2 \in W_x$ we have $w_1 \preceq_x w_2$ if and only if $w_1 \preceq_y w_2$.
- c *Connectedness*: For all $x \in W$, for all $w_1, w_2 \in W_x$ it holds that $w_1 \preceq_x w_2$ or $w_2 \preceq_x w_1$.

In this chapter we introduce labelled proof systems for $\mathbb{PCL}$ and its extensions on the basis of neighbourhood semantics. Neighbourhood models were introduced in Definition 1.4.10 of Chapter 1. Extensions of these models are defined similarly as extensions for sphere models (Definition 1.4.9).

Definition 3.1.3. Extensions of neighbourhood models are defined as follows:

- N *Normality*: For all $x \in W$ it holds that $N(x) \neq \emptyset$;
- T *Total reflexivity*: For all $x \in W$ there exists $\alpha \in N(x)$ such that $x \in \alpha$;
- W *Weak centering*: For all $x \in W$ and $\alpha \in N(x)$, $x \in \alpha$;
- C *Centering*: For all $x \in W$ and $\alpha \in N(x)$, $\{x\} \subseteq \alpha$ and $\{x\} \in N(x)$;
- U *Local Uniformity*: For all $x \in W$ it holds that if $y \in \alpha$ and $\alpha \in N(x)$, then $\bigcup N(x) = \bigcup N(y)$.
- A *Local absoluteness*: For all $x \in W$ it holds that if $y \in \alpha$ and $\alpha \in N(x)$, then $N(x) = N(y)$.
- Nes *Nesting*: For all $\alpha, \beta \in N(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

Generally speaking, the use of classes of models different from Kripke semantics as the basis for the definition of labelled calculi is an established procedure. For instance, Negri and Sbardolini defined in [76] a labelled sequent calculus for $\mathbb{VC}$ by internalising a Kripke semantics equipped with a ternary accessibility relation, this class of models being a special case of preferential semantics. Particularly relevant for our scope is [75], in which Negri and Olivetti defined a labelled sequent calculus for $\mathbb{PCL}$ based on neighbourhood a semantics: here their method is extended to cover all extensions of $\mathbb{PCL}$. Moreover in [71] is defined a general method to define sequent calculus on the basis of a neighbourhood semantics.

In order to define labelled calculi for preferential logics based on neighbourhood models, we need to prove soundness and completeness of the classes of models with respect to the axioms of $\mathbb{PCL}$ and its extensions. An axiomatization of preferential logics can be found in Figure 1.6 of Chapter 1. We denote by $\vdash_{\mathcal{H}^*} F$ derivability of formula F from the axiom systems for $\mathbb{PCL}$ and extensions. Moreover, let P^*-models denote any extension of P-models with the semantic conditions from Definition 3.1.2 and N^*-models denote any extension of N-models with the semantic conditions listed in Definition 3.1.3. With $\Vdash_{N^*} F$ (resp. $\Vdash_{P^*} F$) we mean validity of formula F at a class of neighbourhood (resp. preferential) models.

Theorem 3.1.1 (Soundness)**.** For $F \in \mathcal{F}^{>}$, it holds that if $\vdash_{\mathcal{H}^*} F$ then $\Vdash_{N^*} F$.

Proof. We prove soundness of axioms (CM), (OR), (U_1) and (CV). Proofs of the other axioms can be reconstructed form the derivations of the axioms in the labelled sequent calculi, in Appendix 3.A at the end of this chapter.
(CM) $(A > B) \wedge (A > C) \rightarrow (A \wedge B) > C$. Suppose the antecedent of the axiom is valid in a neighbourhood model, while the conclusion is not. We show that a contradiction is reached. There is a model $\mathcal{M}$ such that:

1. $\mathcal{M}, x \Vdash A > B$, i.e., if there exists $\alpha \in N(x)$ such that $\alpha \Vdash^{\exists} A$, then there exists $\beta \subseteq \alpha$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \rightarrow B$;
2. $\mathcal{M}, x \Vdash A > C$, i.e., if there exists $\alpha \in N(x)$ such that $\alpha \Vdash^{\exists} A$, then there exists $\gamma \subseteq \alpha$ such that $\gamma \Vdash^{\exists} A$ and $\gamma \Vdash^{\forall} A \rightarrow C$;
3. $\mathcal{M}, x \nVdash (A \wedge B) > C$.

From 3 it follows that:

4. There exists $\alpha \in N(x)$ such that $\alpha \Vdash^{\exists} A$;
5. For all $\beta \subseteq \alpha$ it holds that $\beta \Vdash^{\exists} A \wedge B \wedge \neg C$.

From 1 and 4 we obtain 6, and from 2 and 6 we obtain 7:

6. There exists $\beta \subseteq \alpha$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \rightarrow B$;
7. There exists $\gamma \subseteq \beta$ such that $\gamma \Vdash^{\exists} A$ and $\gamma \Vdash^{\forall} A \rightarrow C$.

By transitivity, $\gamma \subseteq \alpha$; thus, from 5 we have that there exists $y \in \gamma$ such that $y \Vdash A \wedge B \wedge \neg C$. From 7, however, for all $y \in \gamma$ it holds that if $y \Vdash A$ then $y \Vdash C$, whence we obtain a contradiction.
(OR) $(A > C) \wedge (B > C) \rightarrow (A \vee B) > C$. Suppose there is a neighbourhood model which satisfies the antecedent and falsifies the conclusion of the implication:

1. $\mathcal{M}, x \Vdash A > B$, i.e., if there exists $\alpha \in N(x)$ such that $\alpha \Vdash^{\exists} A$, then there exists $\beta \subseteq \alpha$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \rightarrow B$;
2. $\mathcal{M}, x \Vdash B > C$, , i.e., if there exists $\alpha \in N(x)$ such that $\alpha \Vdash^{\exists} B$, then there exists $\gamma \subseteq \alpha$ such that $\gamma \Vdash^{\exists} B$ and $\gamma \Vdash^{\forall} B \rightarrow C$;
3. $\mathcal{M}, x \nVdash (A \vee B) > C$.

From 3 we have that:

4. There exists $\alpha \in N(x)$ such that $\alpha \Vdash^{\exists} A \vee B$;
5. For all $\beta \subseteq \alpha$ it holds that $\beta \Vdash^{\exists} (A \vee B) \wedge \neg C$.

Suppose that $\alpha \Vdash^{\exists} A$ (the case for $\alpha \Vdash^{\exists} B$ is similar). From 1, we have:

6. There exists $\beta \subseteq \alpha$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \rightarrow C$.

From 5 and 6 we have that $\beta \Vdash^{\exists} (A \vee B) \wedge \neg C$; thus, there exists $y \in \beta$ such that $y \Vdash A \vee B$ and $y \Vdash C$. By the last conjunct of 6, $y \Vdash A \rightarrow C$; thus, $y \Vdash B$, whence $\beta \Vdash^{\exists} B$. From this latter and 2 we obtain:

7. There exists $\gamma \subseteq \beta$ such that $\gamma \Vdash^{\exists} B$ and $\gamma \Vdash^{\forall} B \rightarrow C$.

By transitivity, $\gamma \subseteq \alpha$; thus, by 3 and 7 we obtain $\gamma \Vdash^{\exists} (A \vee B) \wedge \neg C$, i.e., there exists $z \in \gamma$ such that $z \Vdash A \vee B$ and $z \Vdash \neg C$. Moreover, from 7 we have that $z \Vdash B \rightarrow C$ and from 6 and $\gamma \subseteq \beta$ we have $z \Vdash A \rightarrow C$, and we have reached a contradiction.
(U_1) $\neg A > \bot \rightarrow \neg(\neg A > \bot) > \bot$. Suppose there is a neighbourhood model with local uniformity that verifies the antecedent and falsifies the consequent.

1. $\mathcal{M}, x \Vdash \neg A > \bot$, , i.e., if there exists $\alpha \in N(x)$ such that $\alpha \Vdash^{\exists} \neg A$, then there exists $\beta \subseteq \alpha$ such that $\beta \Vdash^{\exists} \neg A$ and $\beta \Vdash^{\forall} \neg A \rightarrow \bot$;

2. $\mathcal{M}, x \not\Vdash \neg(\neg A > \bot) > \bot$.

From 2 we have that there exists $\alpha \in N(x)$ such that $\alpha \Vdash^{\exists} \neg(\neg A > \bot)$; thus, there exists $y \in \alpha$ such that $y \Vdash \neg(\neg A > \bot)$. From this we have that there exists $\beta \in N(y)$ such that $\beta \Vdash^{\forall} \neg A$ (the second conjunct of the condition is superfluous). Then, there is a $z \in \beta$ such that $z \Vdash \neg A$. By the condition of uniformity, $\bigcup N(x) = \bigcup N(y)$; thus, since $z \in \bigcup N(y)$, $z \in \bigcup N(x)$, and there exists a neighbourhood $\gamma \in N(x)$ such that $z \in \gamma$, and $\gamma \Vdash^{\exists} \neg A$. From this condition and 1 we obtain that there exists $\delta \subseteq \gamma$ such that $\delta \Vdash^{\exists} \neg A$ and $\delta \Vdash^{\forall} \neg A \to \bot$. Thus, there is a world $w \in \delta$ such that $w \Vdash \bot$, and we have a contradiction.
(CV) $(A > C) \wedge \neg(A > \neg B) \to (A \wedge B) > C$. Suppose there is a neighbourhood model in which the condition of nesting holds that satisfies the antecedent but not the consequent of the implication.

1. $\mathcal{M}, x \Vdash A > C$, i.e. , i.e., if there exists $\alpha \in N(x)$ such that $\alpha \Vdash^{\exists} A$, then there exists $\beta \subseteq \alpha$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$;
2. $\mathcal{M}, x \Vdash \neg(A > \neg B)$;
3. $\mathcal{M}, x \not\Vdash (A \wedge B) > C$.

From 2 and 3 we have that:

4. There exists $\alpha \in N(x)$ such that $\alpha \Vdash^{\exists} A$;
5. For all $\beta \subseteq \alpha$ it holds that $\beta \Vdash^{\exists} A \wedge B$;
6. There exists $\beta \in N(x)$ such that $\beta \Vdash^{\exists} A \wedge B$;
7. For all $\gamma \subseteq \beta$ it holds that $\gamma \Vdash^{\exists} A \wedge B \wedge \neg C$.

From 1 and 4 we obtain:

8. There exists $\gamma \subseteq \alpha$ such that $\gamma \Vdash^{\exists} A$ and $\gamma \Vdash^{\forall} A \to C$.

We now apply the condition of nesting of neighbourhoods, and obtain that either $\gamma \subseteq \beta$ or $\beta \subseteq \gamma$. We start with the former case: from $\gamma \subseteq \beta$ and 7 we obtain that $\gamma \Vdash^{\exists} A \wedge B \wedge \neg C$. Thus, there exists $y \in \gamma$ such that $y \Vdash A \wedge B \wedge \neg C$; and from the second conjunct of 8 we have that $y \Vdash A \to C$, and we reach a contradiction. If $\beta \subseteq \gamma$, by reflexivity we obtain $\beta \subseteq \beta$, and from this and 7 we have that $\beta \Vdash^{\exists} A \wedge B \wedge \neg C$. Thus, there is a $z \in \beta$ such that $z \Vdash A \wedge B \wedge \neg C$. From $\beta \subseteq \gamma$ and the second conjunct of 8 we have $z \Vdash A \to C$, whence we get a contradiction. □

Burgess proved completeness of preferential models with respect to the axioms of $\mathbb{PCL}$ [18] and Halpern and Friedman in [28] extended the result to cover the extensions of $\mathbb{PCL}$ with normality, total reflexivity, centering, uniformity, absoluteness and connectedness. Thus, we have the following:

Theorem 3.1.2. For $F \in \mathcal{F}^{>}$, it holds that if $\Vdash_{P^*} F$ then $\vdash_{\mathcal{H}^*} F$.

To the best of our knowledge, no completeness result is known for the axioms of $\mathbb{PCL}$ and its extensions with respect to neighbourhood models. In the case of sphere models, that is to say, nested neighbourhood models, a direct completeness result was given by Lewis in [57, Chapter 6]: he proved that the axioms of $\mathbb{V}$ and extensions are sound and complete with respect to sphere models[1]. We here rely on Burgess' completeness result for preferential semantics and obtain completeness of the axiomatization with respect to neighbourhood models by showing the equivalence between preferential and neighbourhood models.

[1]Sphere models are nested neighbourhood models. Lewis' truth condition of the conditional operator in sphere models is slightly different than the truth condition of the conditional operator we defined for neighbourhood models. However, via the condition of nesting, the two truth conditions are equivalent: refer to Remark 3.3.2.

Remark 3.1.1. Burgess' completeness proof appeared in a 9-page article titled "Quick completeness proofs for some logics of conditionals" [18]. However, the result is quite technical and condensed in a few pages - thus it is not so easy to grasp. In [28] Halpern and Friedman, interested in determining the complexity of conditional logics, detailed a completeness proof for $\mathbb{PCL}$ similar to the one given by Burgess. In their article, they state that the result can be extended to extensions of $\mathbb{PCL}$, but the proof is postponed to a full paper which never appeared. More recently, Giordano, Gliozzi, Olivetti and Schwind proved completeness of the axiomatization of $\mathbb{PCL}$ and its extensions with respect to classes of preferential models with the Limit assumption [34].

In the literature, there are just a few proofs of completeness of preferential logics; thus, it would be interesting to have a direct proof of completeness proof between neighbourhood models and the axiomatic presentation of $\mathbb{PCL}$.

3.2 From preferential models to neighbourhood models (and back)

The result of equivalence between preferential and neighbourhood structures relies on a result by P. Alexandroff, known as duality between partial orders and Alexandroff topologies [7]. In [57, Chapter 2, Section 3] Lewis proved the correspondence between preferential models and sphere models in the case of logic $\mathbb{VC}$[2]. More recently, Marti and Pinosio defined a duality between preferential structures and topological spaces [63]. Negri and Olivetti proved the equivalence between preferential and neighbourhood models in the case of logic $\mathbb{PCL}$ [75]. In the first part of this section, we formalize the correspondence for extensions of preferential and neighbourhood models. To prove one side of the equivalence, we employ an indirect argument. Thanks to this result, and to the proof of completeness with respect to preferential models (Theorem 3.1.2), we obtain completeness of neighbourhood models with respect to the axioms of $\mathbb{PCL}$ and extensions (Theorem 3.2.3).

In the second part of the section we introduce a new class of neighbourhood models: *closed neighbourhood models*. For this class of models the two directions of the equivalence with preferential models can be proved in a constructive way (Theorem 3.2.5). As a consequence, we obtain equivalence of neighbourhood models and closed neighbourhood models (Theorem 3.2.4).

Completeness of neighbourhood models

In order to prove completeness of neighbourhood semantics with respect to the axiom system for $\mathbb{PCL}$ and its extensions, we show the equivalence between preferential models and neighbourhood models, i.e., that for $F \in \mathcal{F}^{>}$ the following holds:

$$\Vdash_{N^*} F \textit{ if and only if} \Vdash_{P^*} F.$$

The [$\mathsf{only\ if}$] direction can be proved directly (Theorem 3.2.1) by constructing neighbourhood models from preferential models. However, it is not possible to define a preferential model from a neighbourhood model. Strictly speaking, this is not a problem, since the [if] direction of the equivalence can be retrieved by means of an indirect argument, employing completeness of the axiomatization of $\mathbb{PCL}$ with respect to preferential models and soundness of neighbourhood models.

Definition 3.2.1. The weight of a formula of $\mathbb{PCL}$ and extensions is defined as follows:

[2]For Lewis' logics, the result can be intuitively understood recalling the fact that systems of spheres (here: nested neighbourhood) are supposed to carry information on the comparative similarity between worlds occurring in the spheres.

- $w(p) = w(\bot) = 0$ for p atomic formula;
- $w(A \circ B) = w(A) + w(B) + 1$ for $\circ \in \{\wedge, \vee, \rightarrow\}$;
- $w(A > B) = w(A) + w(B) + 2$.

Theorem 3.2.1. For $F \in \mathcal{F}^{>}$, if F is valid in the class of N^*-models, then the formula is valid in the class of P^*-models, i.e., if $\Vdash_{N^*} F$ then $\Vdash_{P^*} F$.

Proof. Given a P^*-model, we construct a N^*-model and show that for any formula A, A is valid in the N^*-model if and only if A is valid in the P^*-model. The proof is by induction on the weight of formulas. We first detail the construction in the case of P-models and N-models. Let $\mathcal{M}^P = \langle W, \{W_x\}_{x \in W}, \{\preceq_x\}_{x \in W}, [\![\ ]\!]\rangle$ be a P-model. We define a neighbourhood model $d(\mathcal{M}^N) = \langle W^N, N, [\![\ ]\!]^N\rangle$ as follows:

- $W^N = W$;
- For all $x \in W^N$, $N(x) = \{S \subseteq W_x \mid S \textit{ is downward closed w.r.t. } \preceq_x \textit{ and } S \neq \emptyset\}$, where S is said to be *downward closed with respect to* $\preceq_x$ if, for all $y, z \in W_x$, it holds that if $y \in S$ and $z \preceq_x y$, then $z \in S$.
- $[\![\ ]\!]^N = [\![\ ]\!]$.

We denote by $\downarrow^{\preceq_x} w$ the downward closed set $\{z \in W_x \mid z \preceq_x w\}$. We first show that $d(\mathcal{M}^N)$ satisfies the properties of a neighbourhood model for $\mathbb{PCL}$, i.e., non-emptiness. To prove non-emptiness, let $\alpha \in N(x)$ for some x, α, and let $\alpha = \downarrow^{\preceq_x} w$ for some $w \in W_x$. Since $w \in \downarrow^{\preceq_x} w$, $w \in \alpha$.

To show that $d(\mathcal{M}^N)$ extended with suitable conditions is a N^*-model for the extensions of $\mathbb{PCL}$, we prove it satisfies the following properties.

Normality. If $\mathcal{M}^P$ is a P-model for $\mathbb{PCLN}$ then $d(\mathcal{M}^N)$ is a N-model for $\mathbb{PCLN}$. A P-model for $\mathbb{PCLN}$ satisfies normality, meaning that for all $W_x \neq \emptyset$. Let $w \in W_x$. By definition $\alpha = \downarrow^{\preceq_x} w$ and $w \in$ and $\downarrow^{\preceq_x} w$, thus $w \in \alpha$.

Total reflexivity. If $\mathcal{M}^P$ is a P-model for $\mathbb{PCLT}$ then $d(\mathcal{M}^N)$ is a N-model for $\mathbb{PCLT}$. A P-model for $\mathbb{PCLT}$ satisfies total reflexivity, i.e., for all $x \in W$ it holds that $x \in W_x$. Thus, since $x \in \downarrow^{\preceq_x} x$, by definition it holds that there exists $\alpha \in N(x)$ such that $x \in \alpha$, and total reflexivity holds.

Weak centering. If $\mathcal{M}^P$ is a P-model for $\mathbb{PCLW}$ then $d(\mathcal{M}^N)$ is a N-model for $\mathbb{PCLW}$. A P-model for $\mathbb{PCLW}$ satisfies weak centering: for all $x \in W$, $y \in W_x$, $x \preceq_x y$. Thus, for all $y \in W_x$ it holds that $x \in \downarrow^{\preceq_x} y$. By definition it follows that $x \in \alpha$, for all $\alpha \in N(x)$. Thus, weak centering holds in neighbourhood models. Observe that form $x \in \alpha$ for all $\alpha \in N(x)$, it follows $\{x\} \subseteq \alpha$, for all $\alpha \in N(x)$.

Centering if $\mathcal{M}^P$ is a P-model for $\mathbb{PCLC}$ then $d(\mathcal{M}^N)$ is a N-model for $\mathbb{PCLC}$. From weak centering we have that $\{x\} \subseteq \alpha$, for all $\alpha \in N(x)$. We have to show that $\{x\} \in N(x)$. By the property of centering in P-models, for all w such that for all $y \in W_x$, $w \preceq_x y$, it holds that $x = y$. This means that $x \in \downarrow^{\preceq_x} y$, for all y, and that if some w satisfies the same property, $w = x$. Let us consider $\downarrow^{\preceq_x} x$. It holds that the set contains exactly one element, x; by the above property, for any other element $w \in \downarrow^{\preceq_x} x$, $w = x$. Thus, $\downarrow^{\preceq_x} x = \{x\}$, and by definition $\{x\} \in N(x)$.

Local uniformity. If $\mathcal{M}^P$ is a P-model for $\mathbb{PCLU}$ then $d(\mathcal{M}^N)$ is a N-model for $\mathbb{PCLU}$. If $y \in W_x$ then there exists $\alpha = \downarrow^{\preceq_x} u$ for some $u \in W_x$ such that $y \in \alpha$. Thus, if $y \in W_x$ then $y \in \bigcup N(x)$. The property of uniformity for neighbourhood models immediately follows from this observation and from the fact that $W_x = W_y$.

Local absoluteness. If $\mathcal{M}^P$ is a P-model for $\mathbb{PCLA}$ then $d(\mathcal{M}^N)$ is a N-modelfor $\mathbb{PCLA}$. Let $\alpha \in N(x)$ and $y \in \alpha$. By definition, $\alpha = \downarrow^{\preceq_x} u$ for some $u \in W_x$ and $y \in \downarrow^{\preceq_x} u$, thus $y \preceq_x u$. By local absoluteness, $y \preceq_y u$. Thus, $y \in \downarrow^{\preceq_y} u$, and for $\alpha = \downarrow^{\preceq_y} u$, $\alpha \in N(y)$ and $y \in \alpha$. Since this holds for all $\alpha \in N(x)$ and $y \in \alpha$, we have that $N(x) = N(y)$.

Nesting. If $\mathcal{M}^P$ is a P-model for $\mathbb{V}$ then $d(\mathcal{M}^N)$ is a N-model for $\mathbb{V}$. A P-model for $\mathbb{V}$ satisfies the property of connectedness: for $w_1, w_2 \in W_x$ either $w_1 \preceq_x w_2$ nor $w_2 \preceq_x w_1$. We show that the corresponding P-model for $\mathbb{V}$ satisfies the property of nesting: for all $\alpha, \beta \in N(x)$, either $\alpha \in \beta$ or $\beta \in \alpha$. Let $\alpha, \beta \in N(x)$. Then, $\alpha = \downarrow^{\preceq_x} u$ for some $u \in W_x$ and $\beta = \downarrow^{\preceq_x} v$ for some $v \in W_x$. By local connectedness we have $u \preceq_x v$ or $v \preceq_x u$. This entails $\downarrow^{\preceq_x} u \subseteq \downarrow^{\preceq_x} v$ or $\downarrow^{\preceq_x} v \subseteq \downarrow^{\preceq_x} u$, from which the result follows.

We now prove that for any $x \in W$ and any formula $F \in \mathcal{F}^{>}$ it holds that

$$\mathcal{M}^P, x \Vdash F \text{ iff } d(\mathcal{M}^N), x \Vdash F.$$

where $\mathcal{M}^P$ is any of the P^*-models and $d(\mathcal{M}^N)$ model for the corresponding neighbourhood model, as defined above. The proof is by induction on the weight of the formula; the base case holds by definition, and the propositional cases easily follow by inductive hypothesis. We show the case $F = A > B$ with respect to P-models. To simplify the notation we write $x \Vdash_P A$ instead of $\mathcal{M}^P, x \Vdash A$ and $x \Vdash_N A$ instead of $\mathcal{M}^N, x \Vdash A$. To prove one direction, suppose that $x \Vdash_P A > B$ holds. This means that:

> *For all $w \in W_x$, if $w \Vdash_P A$ then there exists $y \preceq_x w$ such that $y \Vdash_P A$ and for all $z \preceq_x y, z \Vdash_P A \to B$.*

We have to prove for all $\alpha \in N(x)$ it holds that if $\alpha \Vdash^{\exists} A$ then there exists $\beta \in N(x)$ such that $\beta \subseteq \alpha$, $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$. Suppose $w \Vdash_P A$. By definition, $w \in \downarrow^{\preceq_x} u$ for some $u \in W_x$, and let $\alpha = \downarrow^{\preceq_x} u$; it holds that $\alpha \in N(x)$. By inductive hypothesis, $w \Vdash_N A$; thus $\alpha \Vdash^{\exists} A$.

Now suppose there exists $y \preceq_x w$ such that $y \Vdash_P A$ and $\forall z \preceq_x y, z \Vdash_P A \to B$. Let $\beta = \downarrow^{\preceq_x} y$. Let $\beta = \downarrow^{\preceq_x} y$. By definition we have that $\beta \in N(x)$, and since $y \preceq_x w$ and $w \in \alpha$ we have that $\beta \subseteq \alpha$. By inductive hypothesis, since $y \in \beta$ we have that $y \Vdash_N A$, thus $\beta \Vdash^{\exists} A$. Let $z \in \beta$; by definition, $z \preceq_x y$, and by hypothesis $z \Vdash_P A \to B$. Thus, by inductive hypothesis, $z \Vdash_N A \to B$ for all $z \in \beta$, and $\beta \Vdash^{\forall} A \to B$.

To prove the other direction, suppose that $x \Vdash_N A > B$. This means that:

> *For all $\alpha \in N(x)$ it holds that if $\alpha \Vdash^{\exists} A$ then there exists $\beta \in N(x)$ such that $\beta \subseteq \alpha$, $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$.*

We want to prove that for all $w \in W_x$, if $w \Vdash_P A$ then there exists $y \preceq_x w$ such that $y \Vdash_P A$ and $\forall z \preceq_x y, z \Vdash_P A \to B$. Let $\alpha \Vdash^{\exists} A$. By definition, $\alpha = \downarrow^{\preceq_x} u$, for some $u \in W_x$; and for some $w \in \alpha$ it holds that $w \Vdash_N A$, thus $w \in W_x$ and by inductive hypothesis $w \Vdash_P A$.

Now let $\beta \in N(x)$ such that $\beta \subseteq \alpha$, $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$. Let $\beta = \downarrow^{\preceq_x} y$; by inductive hypothesis, $y \Vdash_N A$. Since $\beta \Vdash^{\forall} A \to B$, by definition and inductive hypothesis we have that for all $z \preceq_x y$ it holds that $z \Vdash_P A \to B$.

(End of the proof). To prove that if A is valid in the class of N^*-models then A is valid in the class of P^*-models, suppose A is valid in $d(\mathcal{M}_N)$. Thus, for all $w \in W$ we have $w \Vdash_N A$, and by what we just proved $w \Vdash_P A$, which means that A is valid in $\mathcal{M}_P$.
To prove that if A is valid in the class of P^*-models then A is valid in the class of N^*-models, reason as follows. Given a P^*-model $\mathcal{M}_P$ we build a N^*-model $d(\mathcal{M}_N)$ as shown above. Moreover, if A is valid in $d(\mathcal{M}_N)$, A is valid in $\mathcal{M}_P$. □

We are now ready to prove the equivalence between the classes of P^*-models and N^*-models.

Theorem 3.2.2. For $F \in \mathcal{F}^{>}$, it holds that $\Vdash_{N^*} F$ if and only if $\Vdash_{P^*} F$.

Proof. To prove one direction, suppose $\Vdash_{N^*} F$. We obtain $\Vdash_{P^*} F$ by Theorem 3.2.1. To prove the other direction, we have to show that if a formula is valid in the class of P^*-models, then the formula is valid in the class of N^*-models. Suppose $\Vdash_P F$. By Theorem 3.1.2, we obtain $\vdash_{\mathcal{H}^*} F$ and, by Theorem 3.1.1, $\Vdash_N F$. □

Finally, from Theorem 3.2.2 and Theorem 3.1.2 we obtain completeness the axiomatization of $\mathbb{PCL}$ and its extensions with respect to neighbourhood models.

Corollary 3.2.3 (Completeness)**.** For $F \in \mathcal{F}^>$, if $\Vdash_{N^*} F$ then $\vdash_{\mathcal{H}^*} F$.

Closed neighbourhood models

If we add the property of closure under non-empty intersection to neighbourhood models, it is possible to prove in a constructive way both directions of the correspondence between neighbourhood and preferential models. We here introduce the class of neighbourhood models closed under intersections (CN^*-models) and show that, for $F \in \mathcal{F}^>$, the following holds:

$$\Vdash_{CN^*} F \quad \textit{if and only if} \quad \Vdash_{P^*} F.$$

The [only if] direction is proved similarly as in the neighbourhood case. Moreover, it is now possible to prove the [if] direction by constructing a preferential model on the basis of a closed neighbourhood model (Theorem 3.2.5). From this result, and from Theorem 3.2.2, we obtain equivalence between neighbourhood and closed neighbourhood models.

Definition 3.2.2. A *closed neighbourhood model* (CN-model) is a neighbourhood model to which the following condition is added:

Closure under non-empty intersections: If $S \subseteq N(x)$ and $\bigcap S \neq \emptyset$, then $\bigcap S \in N(x)$.

Extensions of closed neighbourhood models, denoted by CN^*-models, are defined in the same way as extensions of neighbourhood models (Definition 3.1.3). For $F \in \mathcal{F}^>$, let $\Vdash_{CN^*} F$ denote validity of formula F at a class of closed neighbourhood models.

Theorem 3.2.4. For $F \in \mathcal{F}^>$, if F is valid in the class of CN^*-models, then the formula is valid in the class of P^*-models, i.e., if $\Vdash_{CN^*} F$ then $\Vdash_{P^*} F$.

Proof. The proof proceeds in the same way as the proof of Theorem 3.2.1. In addition, we need to show that the model $d(\mathcal{M}_N)$ there defined satisfies the property of closure under non-empty intersections. To this end, let $S \subseteq N(x)$ and $\bigcap S \neq \emptyset$. We have to prove that $\bigcap S \in N(x)$, i.e. that $\bigcap S$ is non-empty (immediate, from the hypothesis) and that $\bigcap S$ is downward closed with respect to $\preceq_x$. Since $S \subseteq N(x)$, all elements of S are downward closed with respect to $\preceq_x$: thus, for all $\alpha \in S$, it holds that

$(*)$ for all $y, z \in W_x$, if $y \in \alpha$ and $z \preceq_x y$, then $z \in \alpha$.

Suppose that, for $y, z \in W_x$, $y \in \bigcap S$ and $z \preceq_x y$. We have to show that $z \in \bigcap S$. From $y \in \bigcap S$ we have that for all $\alpha \in S$, $y \in \alpha$. Consider an arbitrary $\alpha \in S$. Since $y \in \alpha$ and $z \preceq_x y$, from $(*)$ we have $z \in \alpha$. Since this holds for all $\alpha \in S$, we conclude $z \in \bigcap S$. □

Theorem 3.2.5. For $F \in \mathcal{F}^>$, if F is valid in the class of P^*-models, then the formula is valid in the class of CN^*-models, i.e., if $\Vdash_{P^*} F$ then $\Vdash_{CN^*} F$.

Proof. Given a CN^*-model, we define a P^*-model and show that for any formula F, F is valid in the P^*-model if and only if A is valid in the CN^*-model. The proof is by induction on the weight of formulas.

We first detail the construction in the case of P-models and CN-models. Thus, let $\mathcal{M}^{CN} = \langle W, N, [\![\ ]\!]\rangle$ be an CN-model. We construct a P-model $d(\mathcal{M}^P)$ as follows:

- $W^P = W$;
- for all $x \in W^P$, $W_x = \bigcup\{\alpha \mid \alpha \in N(x)\}$;
- for all $x \in W^P$, $y \preceq_x z$ iff for all $\alpha \in N(x)$, if $z \in \alpha$ then $y \in \alpha$;
- $[\![\ ]\!]^P = [\![\ ]\!]$.

We now show that $d(\mathcal{M}^P)$ is a P-model for $\mathbb{PCL}$. This amounts to showing that $\preceq_x$ is reflexive and transitive over W^P. Reflexivity is satisfied, since for all $\alpha \in N(x)$, if $x \in \alpha$ it trivially holds that $x \in \alpha$. As for transitivity, suppose $w \preceq_x k$ and $k \preceq_x z$ hold in $d(\mathcal{M}^P)$. This means that in $\mathcal{M}^N$ the following hold:

a) For all $\alpha \in N(x)$, if $k \in \alpha$ then $w \in \alpha$;
b) For all $\alpha \in N(x)$, if $z \in \alpha$ then $k \in \alpha$.

Let $z \in \alpha$. By b) we get $k \in \alpha$ and by a) $w \in \alpha$. Thus, for all $\alpha \in N(x)$ it holds that if $z \in \alpha$ then $w \in \alpha$, which means that $w \preceq_x z$ holds in $d(\mathcal{M}^P)$.
To show that $d(\mathcal{M}^P)$ extended with suitable conditions is a P^*-model for the extensions of $\mathbb{PCL}$, we have to prove that it satisfies the following properties.

Normality. If $\mathcal{M}^{CN}$ is a CN-model for $\mathbb{PCLN}$ then $d(\mathcal{M}^P)$ is a P-model for $\mathbb{PCLN}$. A CN-model for $\mathbb{PCLN}$ satisfies normality: for all x, there exists $\alpha \in N(x)$. By definition of $d(\mathcal{M}^P)$, this means that W_x is non-empty; thus, normality holds.

Total reflexivity. If $\mathcal{M}^{CN}$ is a CN-model for $\mathbb{PCLT}$ then $d(\mathcal{M}^P)$ is a P-model for $\mathbb{PCLT}$. A CN-model for $\mathbb{PCLT}$ satisfies total reflexivity, i.e., for all x there exists an $\alpha \in N(x)$ such that $x \in \alpha$. By definition of $d(\mathcal{M}^P)$, this yields that $x \in W_x$; thus, $d(\mathcal{M}^P)$ satisfies total reflexivity.

Weak centering. If $\mathcal{M}^{CN}$ is a CN-model for $\mathbb{PCLW}$ then $d(\mathcal{M}^P)$ is a P-model for $\mathbb{PCLW}$. A CN-model for $\mathbb{PCLW}$ satisfies weak centering: each $N(x)$ is non-empty, and for all $\alpha \in N(x)$ it holds that $x \in \alpha$. Let $\alpha \in N(x)$; by the property of non-emptiness of neighbourhood models, there exists $y \in \alpha$. By weak centering $x \in \alpha$. Thus, for all $\alpha \in N(x)$ it holds that if $y \in \alpha$ then $x \in \alpha$, which by definition means $x \preceq_x y$, and weak centering is satisfied.

Centering if $\mathcal{M}^{CN}$ is a CN-model for $\mathbb{PCLC}$ then $d(\mathcal{M}^P)$ is a P-model for $\mathbb{PCLC}$. A CN-model for $\mathbb{PCLC}$ satisfies centering: for all $\alpha \in N(x)$, it holds that $\{x\} \subseteq \alpha$[3]. If $\{x\} \subseteq \alpha$, then $x \in \alpha$. Let $y \in \alpha$. By the same reasoning employed for weak centering, $x \preceq_x y$, for arbitrary α and y. Now suppose there is a $w \in W_x$ such that for all $y \in W_x$, $w \preceq_x y$. By definition of $d(\mathcal{M}^P)$, this means that

$*$) for all $\alpha \in N(x)$, if $y \in \alpha$ then $w \in \alpha$.

By centering on neighbourhood models, $\{x\} \in N(x)$. Let us consider $\{x\}$. It holds that $x \in \{x\}$ and thus, by a) conclude $w \in \{x\}$, meaning $w = x$. Thus, centering is satisfied in $d(\mathcal{M}^P)$.

Local uniformity. If $\mathcal{M}^{CN}$ is a CN-model for $\mathbb{PCLU}$ then $d(\mathcal{M}^P)$ is a P-model for $\mathbb{PCLU}$. A CN-model for $\mathbb{PCLU}$ satisfies uniformity: for all x, it holds that if $y \in \alpha$ and $\alpha \in N(x)$, then $\bigcup N(x) = \bigcup N(y)$. By definition of the model $d(\mathcal{M}^P)$ it immediately follows that $\bigcup W_x = \bigcup W_y$; thus, local uniformity holds.

Local absoluteness. If $\mathcal{M}^{CN}$ is a CN-model for $\mathbb{PCLA}$ then $d(\mathcal{M}^P)$ is a P-model for $\mathbb{PCLA}$. A CN-model for $\mathbb{PCLA}$ satisfies local absoluteness: for all x, it holds that

[3]The requirement of normality is no longer needed, since $\{x\} \subseteq \{x\}$, and $\{x\} \in N(x)$.

if $y \in \alpha$ and $\alpha \in N(x)$, then $N(x) = N(y)$. This implies that $\bigcup N(x) = \bigcup N(y)$, and $\bigcup W_x = \bigcup W_y$ (local uniformity) holds. Moreover, local absoluteness states that the neighbourhood associated to each world are the same; thus, given $x, y \in W$ it holds that if for all $\alpha \in N(x)$ we have that if $w_2 \in \alpha$ then $w_1 \in \alpha$, then also for all $\alpha \in N(y)$ we have that if $w_2 \in \alpha$ then $w_1 \in \alpha$. Thus, if $w_1 \preceq_x w_2$ then $w_1 \preceq_y w_2$; local absoluteness holds.

Connectedness. If $\mathcal{M}^{CN}$ is a CN-model for $\mathbb{V}$ then $d(\mathcal{M}^P)$ is a P-model for $\mathbb{V}$. A CN-model for $\mathbb{V}$ satisfies the property of nesting. We show that the corresponding P-model for $\mathbb{V}$ satisfies the property of strong connectedness. Suppose by contradiction that for $w_1, w_2 \in W_x$ neither $w_1 \preceq_x w_2$ nor $w_2 \preceq_x w_1$. By definition of the model, this means that:

a) There exists $\beta \in N(x)$ such that $w_2 \in \beta$ and $w_1 \notin \beta$;
b) There exists $\gamma \in N(x)$ such that $w_1 \in \gamma$ and $w_2 \notin \gamma$.

By nesting, either $\beta \subseteq \gamma$ or $\gamma \subseteq \beta$. If the former holds, from b) we have $w_2 \in \beta$, and thus $w_2 \in \gamma$, which contradicts with b). If the latter holds, from a) we have $w_1 \in \gamma$, which means $w_1 \in \beta$, that contradicts with a).

We now prove that for any $x \in W$ and any formula $F \in \mathcal{F}^>$ it holds that

$$\mathcal{M}^{CN}, x \Vdash F \text{ iff } d(\mathcal{M}^P), x \Vdash F.$$

This holds for $\mathcal{M}^{CN}$ any of the CN^*-models and $d(\mathcal{M}^P)$ the defined preferential model. The proof is by induction on the weight of F. We only consider the case $F = A > B$, in the class of CN-models. We write $x \Vdash_{CN} A$ instead of $\mathcal{M}^{CN}, x \Vdash A$.

To prove one direction, suppose that $x \Vdash_{CN} A > B$. This means that:

For all $\alpha \in N(x)$ it holds that if $\alpha \Vdash^\exists A$ then there exists $\beta \in N(x)$ such that $\beta \subseteq \alpha$, $\beta \Vdash^\exists A$ and $\beta \Vdash^\forall A \to B$.

We want to prove that for all $w \in W_x$, if $w \Vdash_P A$ then there exists $y \preceq_x w$ such that $y \Vdash_P A$ and for all $z \preceq_x y, z \Vdash_P A \to B$. To this aim, suppose there exists $\alpha \in N(x)$ such that $\alpha \Vdash^\exists A$. This means that there exists $w \in \alpha$ and $w \Vdash_{CN} A$. By definition of the model, there exists $w \in W_x$, and by inductive hypothesis $w \Vdash_P A$.
Let us consider the set

$$\alpha^* = \bigcap\{\alpha \in N(x) \,|\, w \in \alpha\}.$$

It holds that $\alpha^* \neq \emptyset$, since $\alpha \in N(x)$ and $w \in \alpha$. By the property of closure under non-empty intersections, $\alpha^* \in N(x)$. Moreover, it holds that $\alpha^* \Vdash^\exists A$. Thus, by hypothesis there is a $\beta \in N(x)$ such that $\beta \subseteq \alpha^*$, $\beta \Vdash^\exists A$ and $\beta \Vdash^\forall A \to B$. From $\beta \Vdash^\exists A$ we obtain that there is $y \in \beta$ such that $y \Vdash_N A$; by inductive hypothesis, this means that there exists $y \in W_x$ such that $y \Vdash_P A$. To show that $y \preceq_x w$, we need to prove that for all $\gamma \in N(x)$ it holds that if $w \in \gamma$ then $y \in \gamma$. To this aim, suppose there exists $\gamma \in N(x)$ such that $w \in \gamma$. Thus, $\alpha^* \subseteq \gamma$; moreover, since $\beta \subseteq \alpha^*$ we have $\beta \subseteq \gamma$; and since $y \in \beta$, we conclude $y \in \gamma$.

Finally, let $z \preceq_x y$. By definition, this means that for all $\gamma \in N(x)$, if $y \in \gamma$ then $z \in \gamma$. Since $y \in \beta$, we conclude $z \in \beta$; moreover, since by hypothesis $\beta \Vdash^\forall A \to B$, we have $z \Vdash_{CN} A \to B$. By inductive hypothesis, $z \Vdash_P A \to B$.

To prove the other direction, suppose that $x \Vdash_{P^*} A > B$ holds. This means that:

For all $w \in W_x$, if $w \Vdash_P A$ then there exists $y \preceq_x w$ such that $y \Vdash_P A$ and for all $z \preceq_x y$ it holds that $z \Vdash_P A \to B$.

We have to prove for all $\alpha \in N(x)$ it holds that if $\alpha \Vdash^{\exists} A$ then there exists $\beta \in N(x)$ such that $\beta \subseteq \alpha$, $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$. Suppose $w \Vdash_P A$. By definition of the model, this means that there exists $\alpha \in N(x)$ and $w \in \alpha$; by inductive hypothesis, $w \Vdash_{CN} A$; whence $\alpha \Vdash^{\exists} A$.

By hypothesis, there exists $y \preceq_x w$ such that $y \Vdash_P A$. By inductive hypothesis, $y \Vdash_{CN} A$, and by definition there is some $\beta \in N(x)$ such that $y \in \beta$. Let us consider the set

$$\beta^* = \bigcap\{\beta \in N(x) \mid y \in \beta\}.$$

We have that $\beta^* \neq \emptyset$, since $\beta \in N(x)$ and $y \in \beta$. Thus, by closure under non-empty intersections, $\beta^* \in N(x)$. Moreover, by hypothesis $y \preceq_x w$. Thus, for all $\gamma \in N(x)$, if $w \in \gamma$ then $y \in \gamma$. Since $w \in \alpha$, we have that $y \in \alpha$; and thus, $\beta^* \subseteq \alpha$. By inductive hypothesis, since $z \in \beta^*$ we have $\beta^* \Vdash^{\exists} A$.

Finally, by hypothesis we have that for all $z \preceq_x y$ it holds that $z \Vdash_P A \to B$. Thus, for all $\gamma \in N(x)$ we have that if $y \in \gamma$ then $z \in \gamma$. Let $z \in \beta^*$, and let $\gamma \in N(x)$ such that $y \in \gamma$. Thus, $z \in \gamma$, and $\beta^* \subseteq \gamma$. By inductive hypothesis we conclude that $z \Vdash_{CN} A \to B$, and since this holds for all $z \in \beta^*$, we have $\beta^* \Vdash^{\forall} A \to B$.

(End of the proof). To prove that if A is valid in the class of P^*-models then A is valid in the class of N^*-models, suppose A is valid in $d(\mathcal{M}_P)$. Thus, for all $w \in W$ we have $w \Vdash_P A$, and by what we just proved $w \Vdash_N A$, which means that A is valid in $\mathcal{M}_N$.
To prove that if A is valid in the class of N^*-models then A is valid in the class of P^*-models, reason as follows. Given an N^*-model $\mathcal{M}_N$ we build a P^*-model $d(\mathcal{M}_P)$ as shown above. Moreover, if A is valid in $d(\mathcal{M}_P)$, A is valid in $\mathcal{M}_N$. $\square$

From Theorem 3.2.4 and 3.2.5 we obtain equivalence between closed neighbourhood models and preferential models.

Corollary 3.2.6. For $A \in \mathcal{F}^{>}$, it holds that $\Vdash_{CN^*} F$ if and only if $\Vdash_{P^*} F$.

Then, from this result and Theorem 3.2.2 we obtain equivalence between neighbourhood models and closed neighbourhood models.

Corollary 3.2.7. For $A \in \mathcal{F}^{>}$, it holds that $\Vdash_{N^*} F$ if and only if $\Vdash_{CN^*} F$.

3.3 A family of labelled sequent calculi

In the light of the adequacy result of the previous section, we use neighbourhood frames to define a labelled sequent calculus for $\mathbb{PCL}$ and its extensions. The reason to prefer neighbourhood semantics over preferential models relies mainly on the fact that neighbourhood semantics is a generalization of Lewis' sphere semantics: neighbourhoods are non-nested systems of spheres. Thus, neighbourhood semantics allows us to generalise the well-known Lewis' sphere semantics to weaker systems, and provide a uniform treatment for a wide family of conditional logics.

We call **G3CL** the calculus for $\mathbb{PCL}$. Calculi for extensions are denoted by **G3CL** to which we add the name of the frame conditions of the corresponding logics: thus, $\mathbf{G3CL^{N}}$ is a proof system for $\mathbb{PCLN}$, $\mathbf{G3CL^{TU}}$ is a proof system for $\mathbb{PCLTU}$ and $\mathbf{G3CL^{V}}$ is a proof system for $\mathbb{V}$. We denote by $\mathbf{G3CL}^*$ the whole family of calculi.

The definition of the sequent calculi $\mathbf{G3CL}^*$ follows the well-established methodology of enriching the language of the calculus by means of labels, thus importing the semantic information into the syntactic proof system[4]. For this reason, it is useful to recall the the truth condition for the conditional operator in neighbourhood models:

[4]Refer to [69] for the general methodology in Kripke models and to [71] for the general methodology in neighbourhood semantics.

$(*)\ x \Vdash A > B$ *iff* *for all* $\alpha \in N(x)$ *if* $\alpha \Vdash^{\exists} A$ *then there exists* $\beta \in N(x)$ *such that* $\beta \subseteq \alpha$ *and* $\beta \Vdash^{\exists} A$ *and* $\beta \Vdash^{\forall} A \to B$.

We enrich the set $\mathcal{F}^{>}$ as follows.

Definition 3.3.1. Let $x, y, z, \ldots$ be variables for worlds in a neighbourhood model, and $a, b, c, \ldots$ variables for neighbourhoods. *Relational atoms* are formulas of the following form and meaning:

- $a \in N(x)$, "neighbourhood a belongs to the system of neighbourhood associated to x";
- $x \in a$, "world x is included in neighbourhood a";
- $a \subseteq b$, "neighbourhood a is included into neighbourhood b".

Labelled formulas are defined as follows, for $A \in \mathcal{F}^{>}$:

- Relational atoms are labelled formulas;
- $x : A$, "formula A is true at world x";
- $a \Vdash^{\exists} A$, "A is true at some world belonging to neighbourhood a";
- $a \Vdash^{\forall} A$, "A is true at all worlds belonging to neighbourhood a";
- $x \Vdash_a A|B$, "there exists $\beta \in N(x)$ such that $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$".

We shall also use $\{x\}$ to denote a neighbourhood consisting of exactly one element (a singleton).

Relational atoms and labelled formulas are defined in correspondence with semantic notions. Relational atoms describe the structure of the neighbourhood model, whereas labelled formulas are defined in correspondence with the forcing relations at a world ($x \Vdash A$) and at a neighbourhood ($a \Vdash^{\exists} A$, $a \Vdash^{\forall} A$). Formula $x \Vdash_a A|B$ introduces a semantic condition corresponding to the right-hand side of the consequent of $(*)$. The reason for the introduction of this formula is that $(*)$ is too rich to be expressed by a single rule, since it consists of an existential quantifier under the scope of a universal quantifier. Thus we need to break $(*)$ into two smaller conditions, one (the antecedent) covered by rules for formulas $x : A > B$ and the other (the consequent) covered by $x \Vdash_a A|B$.

Definition 3.3.2. Sequents of $\mathbf{G3CL}^{*}$ are expressions $\Gamma \Rightarrow \Delta$ where Γ and Δ are multisets of relational atoms and labelled formulas, and relational atoms may occur only in Γ.

Figure 3.1 contains the rules for $\mathbb{PCL}$, whereas Figure 3.2 shows the rules for extensions of $\mathbb{PCL}$. We write $(\mathsf{a!})$ as a side condition expressing the requirement that label a should not occur in the conclusion of a rule. Propositional rules are standard. Rules for local forcing explicit the meaning of the forcing relations $\Vdash^{\forall}$ and $\Vdash^{\exists}$. Rules for the conditional are defined on the basis of the truth condition for $>$ in neighbourhood models.

Each rule of Figure 3.2 is defined in correspondence with the frame conditions on extensions of $\mathbb{PCL}$. In some cases (rules for total reflexivity and weak centering) the frame condition can be formalized by means of a single rule. Rule $\mathsf{0}$ stands for the requirement of non-emptiness in the model, and it is added to capture the condition of normality, along with rule N[5]. The rule has a detailed side condition $(\star)$, requiring that there should not be any formulas of the form $w \in a$ in Γ. This condition is needed

[5] The rule needs not to be added to the calculus **G3CL**: the rules of this calculus always introduce non-empty neighbourhoods, and the system can be shown to be complete with respect to the axioms of $\mathbb{PCL}$ (Theorem 3.4.7). However, the rule is needed to express the condition of nesting: the new neighbourhood introduced by rule N could be empty.

Initial sequents

$$\frac{}{x : p, \Gamma \Rightarrow \Delta, x : p}\ \mathsf{init} \qquad \frac{}{x : \bot, \Gamma \Rightarrow \Delta}\ \bot_{\mathsf{L}}$$

Rules for local forcing

$$\frac{x : A, x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ \mathsf{L} \Vdash^{\forall} \qquad \frac{x \in a, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\ \mathsf{R} \Vdash^{\forall} \ (\mathsf{x!})$$

$$\frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ \mathsf{L} \Vdash^{\exists} \ (\mathsf{x!}) \qquad \frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{R} \Vdash^{\exists}$$

Propositional rules Standard - Refer to Figure 2.3

Rules for the conditional

$$\frac{a \in N(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}{\Gamma \Rightarrow \Delta, x : A > B}\ \mathsf{R} > (\mathsf{a!})$$

$$\frac{a \in N(x), x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), x : A > B, \Gamma \Rightarrow \Delta}\ \mathsf{L} >$$

$$\frac{c \in N(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, c \Vdash^{\exists} A \quad c \in N(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, c \Vdash^{\forall} A \to B}{c \in N(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}\ \mathsf{R|}$$

$$\frac{c \in N(x), c \subseteq a, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B, \Gamma \Rightarrow \Delta}{x \Vdash_a A|B, \Gamma \Rightarrow \Delta}\ \mathsf{L|} \ (\mathsf{c!})$$

Rules for inclusion

$$\frac{a \subseteq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{Ref} \qquad \frac{c \subseteq a, c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}{c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}\ \mathsf{Tr} \qquad \frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ \mathsf{L} \subseteq$$

Figure 3.1: Sequent calculus **G3CL**

to ensure termination in root-first proof-search: otherwise rule 0 in combination with rule N could create a loop, introducing always new world and neighbourhood labels. Moreover, ($\star$) requires that there should be at least one formula $a \Vdash^{\exists} A$ or $a \Vdash^{\forall} A$ in $\Gamma \cup \Delta$. This ensure that 0 is applied only in the cases it is actually needed - for instance, when there is a formula $a \Vdash^{\exists} A$ in Δ and no formula $w \in a$ in Γ.

Condition of (strong) centering is represented by means of four rules: Rule Single introduces formulas with neighbourhood label $\{x\}$ (the singleton), and the other three rules detail the conditions that such a neighbourhood should satisfy. Rule C ensures that the singleton contains at least one element, and rules Repl_1 and Repl_2 that it contains at most one element: if there is another element $y \in \{x\}$, then the properties holding for x hold also for y (i.e. x and y are the same element).

Similarly, extensions with local uniformity and absoluteness are defined by adding multiple rules. Rules U_1 and U_2 encode the semantic condition of local uniformity. In order to avoid the symbol $\bigcup$ in the sequent language, the rules translate the following two equivalent conditions:

U_1 *Local uniformity*, 1: if there exists $\alpha \in N(x)$ such that $y \in \alpha$ and there exists $\beta \in N(y)$ such that $z \in \beta$, then there exists $\gamma \in N(x)$ such that $z \in \gamma$;

U_2 *Local uniformity*, 2: if there exists $\alpha \in N(x)$ such that $y \in \alpha$ and there exists $\beta \in N(y)$ such that $z \in \beta$, then there exists $\gamma \in N(y)$ such that $z \in \gamma$.

As for local absoluteness, rules A_1 and A_2 encode the information that for any $x \in W$, given $a \in N(x)$ and $y \in a$, if $\beta \in N(x)$ then $\beta \in N(y)$ (rule A_1), and if $\beta \in N(y)$, then $\beta \in N(x)$ (rule A_2). Thus, $N(x) = N(y)$.

Rules for extensions

$$\frac{a \in N(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{N}\ (\mathsf{a!}) \qquad \frac{y \in a, a \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{0}\ (\mathsf{y!})(\star)$$

$$\frac{x \in a, a \in N(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{T}\ (\mathsf{a!}) \qquad \frac{x \in a, a \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{W}$$

$$\frac{x \in \{x\}, \{x\} \in N(x), \Gamma \Rightarrow \Delta}{\{x\} \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{Single} \qquad \frac{\{x\} \in N(x), \{x\} \subseteq a, a \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{C}$$

$$\frac{y \in \{x\}, At(x), At(y), \Gamma \Rightarrow \Delta}{y \in \{x\}, At(x), \Gamma \Rightarrow \Delta}\ \mathsf{Repl}_1\ (*) \qquad \frac{y \in \{x\}, At(x), At(y), \Gamma \Rightarrow \Delta}{y \in \{x\}, At(y), \Gamma \Rightarrow \Delta}\ \mathsf{Repl}_2\ (*)$$

$$\frac{z \in c, c \in N(x), a \in N(x), y \in a, b \in N(y), z \in b, \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, b \in N(y), z \in b, \Gamma \Rightarrow \Delta}\ \mathsf{U}_1\ (\mathsf{c!})$$

$$\frac{z \in c, c \in N(y), a \in N(x), y \in a, b \in N(y), z \in b, \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, b \in N(x), z \in b, \Gamma \Rightarrow \Delta}\ \mathsf{U}_2\ (\mathsf{c!})$$

$$\frac{b \in N(y), a \in N(x), y \in a, b \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, b \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{A}_1 \qquad \frac{b \in N(x), a \in N(x), y \in a, b \in N(y), \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, b \in N(y), \Gamma \Rightarrow \Delta}\ \mathsf{A}_2$$

$$\frac{a \subseteq b, a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta \quad b \subseteq a, a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{Nes}$$

Rules obtained by closure conditions

$$\frac{z \in c, c \in N(x), a \in N(x), x \in a, z \in a, \Gamma \Rightarrow \Delta}{a \in N(x), x \in a, z \in a, \Gamma \Rightarrow \Delta}\ \mathsf{U_1}^* \qquad \frac{a \in N(y), a \in N(x), y \in a, \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, \Gamma \Rightarrow \Delta}\ \mathsf{A_1}^*$$

$$\frac{y \in c, c \in N(x), a \in N(x), y \in a, a \in N(y), \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, a \in N(y), \Gamma \Rightarrow \Delta}\ \mathsf{U_1}^{**}$$

$$\frac{y \in c, c \in N(y), a \in N(x), y \in a, a \in N(y), \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, \Gamma \Rightarrow \Delta}\ \mathsf{U_2}^{**}$$

($\star$) There are no formulas $w \in a$ in Γ; at least one formula $a \Vdash^{\exists} A$ or $a \Vdash^{\forall} A$ occurs in $\Gamma \cup \Delta$.
($*$) $At(x) := x : P, x \in a, a \in N(x), y \in \{x\}$, for P atomic formula.

$$\mathbf{G3CL^N} = \mathbf{G3CL} + \mathsf{N} + \mathsf{0};\ \mathbf{G3CL^T} = \mathbf{G3CL^N} + \mathsf{T};\ \mathbf{G3CL^W} = \mathbf{G3CL^T} + \mathsf{W};$$
$$\mathbf{G3CL^N} = \mathbf{G3CL} + \mathsf{N} + \mathsf{0};\ \mathbf{G3CL^T} = \mathbf{G3CL^N} + \mathsf{T};\ \mathbf{G3CL^W} = \mathbf{G3CL^T} + \mathsf{W};$$
$$\mathbf{G3CL^C} = \mathbf{G3CL^W} + \mathsf{C} + \mathsf{Single} + \mathsf{Repl}_1 + \mathsf{Repl}_2;$$
$$\mathbf{G3CL^U} = \mathbf{G3CL} + \mathsf{U}_1 + \mathsf{U}_2;\ \mathbf{G3CL^{NU/TU/WU/CU}} = \mathbf{G3CL^{N/T/W/C}};$$
$$\mathbf{G3CL^A} = \mathbf{G3CL} + \mathsf{A}_1 + \mathsf{A}_2;\ \mathbf{G3CL^{NA/TA/WA/CA}} = \mathbf{G3CL^{N/T/W/C}} + \mathsf{A}_1 + \mathsf{A}_2;$$
$$\mathbf{G3CL^V} = \mathbf{G3CL} + \mathsf{Nes};\ \mathbf{G3CL^{VN/VT/VW/VC}} = \mathbf{G3CL^{N/T/W/C}} + \mathsf{Nes};$$
$$\mathbf{G3CL^{VU}} = \mathbf{G3CL^U} + \mathsf{Nes};\ \mathbf{G3CL^{VNU/VTU/VWU/VCU}} = \mathbf{G3CL^{NU/TU/WU/CU}} + \mathsf{Nes};$$
$$\mathbf{G3CL^{VA}} = \mathbf{G3CL^A} + \mathsf{Nes};\ \mathbf{G3CL^{VNA/VTA/VWA/VCA}} = \mathbf{G3CL^{NA/TA/WA/CA}} + \mathsf{Nes}.$$

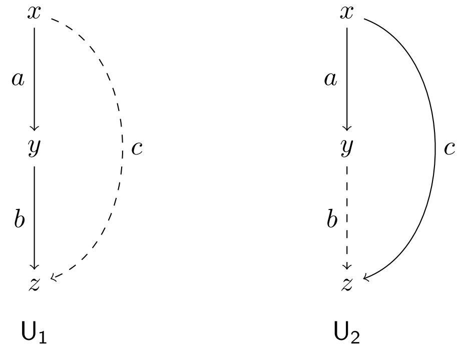
Figure 3.2: Sequent calculi for extensions of **G3CL**

It might happen that some instances of rules of $\mathbf{G3CL}^*$ present a duplication of the atomic formula in the conclusion: for example, an instance of U_1 with $a = b$ displays two formulas $a \in N(x)$ in the conclusion. Since we want contraction to be height-preserving admissible, we deal with these cases by adding to the sequent calculus a new rule, in which the duplicated formulas are contracted into one. Such an operation is called applying a *closure condition* to the rules. Thus, rule $\mathsf{U_1}^*$ is the rule obtained applying the closure condition to U_1 in case $a = b$ and $x = y$; rules $\mathsf{U_1}^{**}$ and $\mathsf{U_2}^{**}$ are obtained from U_1 and U_2, in case $a = b$ and $y = z$; and finally, $\mathsf{A_1}^*$ is obtained from A_1 in the case $a = b$. There is no need to define the additional rules which can be generated by the closure condition, since such rules either collapse or are subsumed by other rules of the calculus. For instance, the rule obtained applying the closure condition to U_2, case $a = b$ and $x = y$, is the following:

$$\frac{z \in c, c \in N(x), a \in N(x), x \in a, \Gamma \Rightarrow \Delta}{a \in N(x), x \in a, \Gamma \Rightarrow \Delta}\ \mathsf{U_2}^*$$

and this is the same instance we obtain applying the closure condition to $\mathsf{U_1}^*$. However, the rules added by closure condition are not needed to prove completeness of the calculi; for this reason, we have not included them in the following sections (e.g. in the termination proof).

Remark 3.3.1. If we represent the two relations $a \in N(x)$ and $y \in a$ as $x \xrightarrow{a} y$, the rules U_1 and U_2 can be respectively represented by means of the following graphs, in which the dashed lines represent the informations added in the premiss of the rule.



The left-hand side graph corresponds to the property of transitivity over worlds x, y and z, while the right-hand side graph corresponds to the property of Euclideaness between the same elements[6]. Moreover, rule U_1 is needed to derive axiom (U_1), and rule U_2 is needed to derive axiom (U_2). Thus, axiom (U_1) imposes transitivity over the worlds of a neighbourhood model, while axiom (U_2) imposes Euclideaness. Not unsurprisingly, recalling that $\Box A = \neg A > \bot$ (Section 1.6), we have that (U_1) corresponds to axiom (4) of modal logics, valid in all Kripke models with transitive frames. Similarly, (U_2) corresponds to axiom (5) of modal logic, valid in all models with Euclidean frames.

$(\mathrm{U}_1)\ \ (\neg A > \bot) \to \neg(\neg A > \bot) > \bot$	$(4)\ \ \Box A \to \Box\Box A$
$(\mathrm{U}_2)\ \ \neg(A > \bot) \to ((A > \bot) > \bot)$	$(5)\ \ \Diamond A \to \Box\Diamond A$

In fact, logic **K45** is the outer modal logic of conditional logics with uniformity: the $\Box$ operator defined in these systems corresponds to the $\Box$ operator of **K45**. Similarly, axiom (T) $A \to \neg(A > \bot)$ of conditional logics corresponds to the axiom of reflexivity (T) $A \to \Box A$ of modal logics. Thus, the outer modal logics of conditional systems with total reflexivity and uniformity is **S5**.

Remark 3.3.2. In models that display the condition of nesting of neighbourhood (i.e., *sphere models*), the truth condition for the conditional $(*)$ can be stated in a simpler way $(**)$.

$(*)$ $x \Vdash A > B$ *iff* *for all* $\alpha \in N(x)$ *if* $\alpha \Vdash^{\exists} A$ *then there exists* $\beta \in N(x)$ *such that* $\beta \subseteq \alpha$ *and* $\beta \Vdash^{\exists} A$ *and* $\beta \Vdash^{\forall} A \to B$.

$(**)$ $x \Vdash A > B$ *iff* *for all* $\alpha \in N(x)$ *if* $\alpha \Vdash^{\exists} A$ *then there exists* $\beta \in N(x)$ *such that* $\beta \Vdash^{\exists} A$ *and* $\beta \Vdash^{\forall} A \to B$.

[6] The property of Eculideaness states that for all x, y, z, it holds that if xRy and xRz then yRz. A Kripke frame has this property if and only if the axiom (5) is valid in the model.

Condition $(**)$ corresponds to Lewis' truth condition for the counterfactual conditional. The two conditions are equivalent. The direction from $(*)$ to $(**)$ is trivial. Conversely, assume $(**)$. Suppose $\alpha \Vdash^{\exists} A$ holds. By nesting, we have that $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$, and thus $\alpha \cap \beta = \alpha$ or $\alpha \cap \beta = \beta$. In both cases we take $\beta' \equiv \alpha \cap \beta$, and we have that $\beta \subseteq \alpha$, $\beta' \Vdash^{\exists} A$. Since $\beta \Vdash^{\forall} A \to B$, a fortiori $\beta' \Vdash^{\forall} A \to B$.
Thus, in systems with nesting the rules for the conditional can be given in a simplified form: there is no need to specify the inclusion of one neighbourhood into the other.

$$\frac{a \in N(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash A|B}{\Gamma \Rightarrow \Delta, x : A > B} \; \mathsf{R} >_{\mathsf{N}} \text{ (a!)}$$

$$\frac{a \in N(x), x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad x \Vdash A|B, a \in N(x), x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), x : A > B, \Gamma \Rightarrow \Delta} \; \mathsf{L} >_{\mathsf{N}}$$

$$\frac{a \in N(x), \Gamma \Rightarrow \Delta, x \Vdash A|B, a \Vdash^{\exists} A \quad a \in N(x), \Gamma \Rightarrow \Delta, x \Vdash A|B, a \Vdash^{\forall} A \to B}{a \in N(x), \Gamma \Rightarrow \Delta, x \Vdash A|B} \; \mathsf{R} >_{\mathsf{N}}$$

$$\frac{a \in N(x), a \Vdash^{\exists} A, a \Vdash^{\forall} A \to B, \Gamma \Rightarrow \Delta}{x \Vdash A|B, \Gamma \Rightarrow \Delta} \; \mathsf{L} >_{\mathsf{N}} \text{ (a!)}$$

It can be easily verified that the two versions of the calculi are equivalent, via Ref.

To prove soundness of the rules with respect to the corresponding system of logics, we need to interpret relational atoms and labelled formulas in neighbourhood models. Observe that the calculi do not have a direct formula interpretation in the language of $\mathcal{F}^{>}$; but we can define an interpretation for them in neighbourhood models. The notion of realization interprets the labels in neighbourhood frames, thus connecting the syntactic elements of the calculus with the semantic elements of the model.

Definition 3.3.3. Let $\mathcal{M} = \langle W, N, [\![\;]\!] \rangle$ be a neighbourhood model for $\mathbb{PCL}$ or its extensions, $\mathcal{S}$ a set of world labels and $\mathcal{N}$ a set of neighbourhood labels. An $\mathcal{SN}$-*realization* over $\mathcal{M}$ consist of a pair of functions (ρ, σ) such that:

- $\rho : \mathcal{S} \to W$: the function assigns to each $x \in \mathcal{S}$ an element $\rho(x) \in W$;
- $\sigma : \mathcal{N} \to \mathcal{P}(W)$: the function assigns to each $a \in \mathcal{N}$ a neighbourhood $\sigma(a) \in N(w)$, for $w \in W$.

We introduce the notion of satisfiability of a formula $\mathcal{F}$ under an $\mathcal{SN}$-realization by cases on the form of $\mathcal{F}$:

- $\mathcal{M} \models_{\rho,\sigma} a \in N(x)$ if $\sigma(a) \in N(\rho(x))$;
- $\mathcal{M} \models_{\rho,\sigma} a \subseteq b$ if $\sigma(a) \subseteq \sigma(b)$;
- $\mathcal{M} \models_{\rho,\sigma} y \in \{x\}$ if $\rho(y) \in \{\sigma(x)\}$;
- $\mathcal{M} \models_{\rho,\sigma} x : P$ if $\rho(x) \Vdash P$ [7];
- $\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\forall} A$ if $\sigma(a) \Vdash^{\forall} A$;
- $\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\exists} A$ if $\sigma(a) \Vdash^{\exists} A$;
- $\mathcal{M} \models_{\rho,\sigma} x \Vdash_a A|B$ if $\sigma(a) \in N(\rho(x))$ and for some $\beta \subseteq \sigma(a)$ it holds that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$;
- $\mathcal{M} \models_{\rho,\sigma} x : A > B$ if for all $\sigma(a) \in N(\rho(x))$, if $\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\exists} A$ then $\mathcal{M} \models_{\rho,\sigma} x \Vdash_a A|B$.

Given a sequent $\Gamma \Rightarrow \Delta$, let $\mathcal{S}, \mathcal{N}$ be the sets of world and neighbourhood labels occurring in $\Gamma \cup \Delta$, and let (ρ, σ) be an $\mathcal{SN}$-realization. Define $\mathcal{M} \models_{\rho,\sigma} \Gamma \Rightarrow \Delta$ if either $\mathcal{M} \not\models_{\rho,\sigma} F$ for some $F \in \Gamma$ or $\mathcal{M} \models_{\rho,\sigma} G$ for some $G \in \Delta$. Define validity under all realizations by $\mathcal{M} \models \Gamma \Rightarrow \Delta$ if $\mathcal{M} \models_{\rho,\sigma} \Gamma \Rightarrow \Delta$ for all (ρ, σ) and say that a sequent is valid in all neighbourhood models if $\mathcal{M} \models_{\rho,\sigma} \Gamma \Rightarrow \Delta$ for all models $\mathcal{M}$.

[7] This definition is extended in the standard way to formulas obtained by the classical propositional connectives.

Theorem 3.3.1 (Soundness)**.** If a sequent $\Gamma \Rightarrow \Delta$ is derivable in **G3CL***, then it is valid in the corresponding class of neighbourhood models.

Proof. The proof is by straightforward induction on the height of the derivation, employing the notion of realization defined before. By means of example, we show soundness of the left and right rule for the conditional operator.

[$\mathsf{L} >$] From a neighbourhood model and a realization which validates the premisses we construct a neighbourhood model which validates the conclusion. Let $\mathcal{M} \vDash_{\rho,\sigma} a \in N(x), x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$ and $\mathcal{M} \vDash_{\rho,\sigma} x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma \Rightarrow \Delta$. The only relevant case is the one in which $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ and $\mathcal{M} \nvDash_{\rho,\sigma} x \Vdash_a A|B$. From the former we have that $\sigma(a) \Vdash^{\exists} A$; from the latter that for $\sigma(a) \in \rho(x)$ and for all $\beta \in \sigma(\alpha)$ it holds that either $\beta \nVdash^{\exists} A$ or $\beta \nVdash^{\forall} A \to B$. By definition, this means that $\mathcal{M} \nvDash_{\rho,\sigma} x : A > B$, for $\sigma(a) \in \rho(x)$; and thus, $\mathcal{M} \vDash_{\rho,\sigma} a \in N(x), x : A > B, \Gamma \Rightarrow \Delta$.

[$\mathsf{R} >$] Suppose $\mathcal{M} \vDash_{\rho,\sigma} a \in N(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B$. We show that the conclusion is valid in the same model, under the same realization. There are two relevant cases: either the one in which $\mathcal{M} \nvDash_{\rho,\sigma} a \Vdash^{\exists} A$ or the one in which $\mathcal{M} \vDash_{\rho,\sigma} x \Vdash_a A|B$. In the former case we have that $\sigma(a) \nVdash^{\exists} A$, for $\sigma(a) \in \rho(x)$. In the latter case, we have that for $\sigma(a) \in \rho(x)$, there exists $\beta \in \sigma(\alpha)$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$. In both cases it holds by definition that $\mathcal{M} \vDash_{\rho,\sigma} x : A > B$; thus, $\mathcal{M} \vDash_{\rho,\sigma} \Gamma \Rightarrow \Delta, x : A > B$. □

3.4 Structural properties

The structural properties of **G3CL** were examined in [75]. In this section we extend the results to the whole family of calculi **G3CL***. We start with some preliminary definitions and lemmas. By the *height* of a derivation we mean the number of nodes occurring in the longest derivation branch, minus one; thus, a derivation consisting only of an initial sequent has height 0. We write $\vdash^n \Gamma \Rightarrow \Delta$ meaning that there is a derivation of $\Gamma \Rightarrow \Delta$ in **G3CL*** with height bounded by n. Following [75], we define an appropriate notion of *weight* of labelled formulas.

Definition 3.4.1. The weight of relational atoms is 0. As for the other labelled formulas, the label of formulas of the form $x : A$ and $x \Vdash_a A|B$ is x; the label of formulas $a \Vdash^{\forall} A$ and $a \Vdash^{\exists} A$ is a. We denote by $l(\mathcal{F})$ the label of a formula $\mathcal{F}$, and by $p(\mathcal{F})$ the pure part of the formula, i.e., the part of the formula without the label and without the forcing relation. The *weight* of a labelled formula is defined as a lexicographically ordered pair

$$\langle w(p(\mathcal{F})), w(l(\mathcal{F}))\rangle$$

where

- for all world labels x, $w(x) = 0$;
- for all neighbourhood labels a, $w(a) = 1$;
- $w(P) = w(\bot) = 1$;
- $w(A \circ B) = w(A) + w(B) + 1$ for $\circ$ conjunction, disjunction or implication;
- $w(A|B) = w(A) + w(B) + 2$;
- $w(A > B) = w(A) + w(B) + 3$.

The definition of substitution of labels given in [69] can be extended in an obvious way to the relational atoms and labelled formulas of **G3CL***. According to this definition we have, for example, $(a \Vdash^{\exists} A)[b/a] \equiv b \Vdash^{\exists} A$, and $(x \Vdash_a B|A)[y/x] \equiv y \Vdash_a B|A$. The calculus is routinely shown to enjoy the property of height preserving substitution both of world and neighbourhood labels:

Proposition 3.4.1.

(i) If $\vdash^n \Gamma \Rightarrow \Delta$, then $\vdash^n \Gamma[y/x] \Rightarrow \Delta[y/x]$;
(ii) If $\vdash^n \Gamma \Rightarrow \Delta$, then $\vdash^n \Gamma[b/a] \Rightarrow \Delta[b/a]$.

Proof. By induction on the height of the derivation. If it is 0, then $\Gamma \Rightarrow \Delta$ is an initial sequent or a conclusion of $\bot_{\mathsf{L}}$. The same then holds for $\Gamma[y/x] \Rightarrow \Delta[y/x]$ and for $\Gamma[b/a] \Rightarrow \Delta[b/a]$. If the derivation has height $n > 0$, we consider the last rule applied. If $\Gamma \Rightarrow \Delta$ has been derived by a rule without variable conditions, we apply the inductive hypothesis and then the rule. Rules with variable conditions require some care in case the substituted variable coincides with the fresh variable in the premiss. This is the case for the rules $\mathsf{R} \Vdash^{\forall}$, $\mathsf{L} \Vdash^{\exists}$, $\mathsf{L}|$, $\mathsf{R} >$, $\mathsf{0}$, N, T, $\mathsf{U_1}$ and $\mathsf{U_2}$. So, if $\Gamma \Rightarrow \Delta$ has been derived by any of these rules, we apply the inductive hypothesis twice to the premiss: the first application replaces the fresh variable with another fresh variable different from the one we want to substitute, and the second occurrence applies the substitution. □

Lemma 3.4.2. The following sequents are derivable in **G3CL***.

1. $a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$
2. $a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A$
3. $x \Vdash_a A|B, \Gamma \Rightarrow \Delta, x \Vdash_a A|B$
4. $x : A, \Gamma \Rightarrow \Delta, x : A$

Proof. The four cases are proved by mutual induction, measuring the weight of labelled formulas. The idea is to apply the left and right rule which correspondingly treat the two identical formula occurrences, until two identical formula occurrences of a smaller weight are reached. All cases are straightforward; by means of example, we show the case 1 and case 4, subcase $x : A > B$. For the former, we have:

$$\cfrac{\cfrac{x \in a, x : A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A, x : A}{x \in a, x : A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{R} \Vdash^{\exists}}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{L} \Vdash^{\exists}$$

Since $w(x : A) < w(a \Vdash^{\exists} A)$, the upper sequent is derivable by inductive hypothesis. For the latter, we proceed similarly by application of the right and left rule:

$$\cfrac{\cfrac{(1) \qquad\qquad (2)}{a \in N(x), a \Vdash^{\exists} A, x : A > B, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}\ \mathsf{L} >}{x : A > B, \Gamma \Rightarrow \Delta, x : A > B}\ \mathsf{R} >$$

where (1) is sequent $a \in N(x), a \Vdash^{\exists} A, x : A > B, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, a \Vdash^{\exists} A$ and (2) is sequent $x \Vdash_a A|B, a \in N(x), a \Vdash^{\exists} A, x : A > B, \Gamma \Rightarrow \Delta, x \Vdash_a A|B$. Both are derivable by inductive hypothesis, since $w(a \Vdash^{\exists} A) < w(x : A > B)$ and $w(x \Vdash_a A|B) < w(x : A > B)$. □

For the following lemmas and theorem, let $\mathcal{F}$ be either a relational atom or a labelled formula. These lemmas establish admissibility of weakening, invertibility and admissibility of contraction, needed to prove admissibility of cut.

Lemma 3.4.3. The rules of weakening are height-preserving admissible in **G3CL***:

$$\cfrac{\Gamma \Rightarrow \Delta}{\mathcal{F}, \Gamma \Rightarrow \Delta}\ \mathsf{Wk_L} \qquad \cfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathcal{F}}\ \mathsf{Wk_R}$$

Proof. The proof amounts to show that if $\vdash^n \Gamma \Rightarrow \Delta$, then $\vdash^n \mathcal{F}, \Gamma \Rightarrow \Delta$ and $\vdash^n \Gamma \Rightarrow \Delta, \mathcal{F}$. The proof is by induction on the height of the derivation and it is straightforward by inductive hypothesis. □

Lemma 3.4.4. All the rules of **G3CL*** are height-preserving invertible.

Proof. Invertibility of the rules means that, given any rule (R), if the conclusion of (R) is derivable with derivation height n, its premiss(es) are derivable with at most the same derivation height. The proof proceeds by straightforward induction on the height of the derivation. We show only a couple of cases. Suppose (R) $= \mathsf{R} \Vdash^{\exists}$, and $\vdash^n x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$. By weakening, $\vdash^n x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A, x : A$. Invertibility of all the rules for the extensions is ensured by weakening.
Suppose (R) $= \mathsf{L} \Vdash^{\exists}$, and $\vdash^n a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta$. In case $n = 0$, the sequent is an initial sequent; then, also $\vdash^n x \in a, x : A, \Gamma \Rightarrow \Delta$ is an initial sequent. If $n > 0$, the sequent has been derived by some rule. If the rule is $\mathsf{L} \Vdash^{\exists}$, then $\vdash^{n-1} x \in a, x : A, \Gamma \Rightarrow \Delta$ and we are done. If the sequent has been derived by some other rule, apply the inductive hypothesis to the premiss and the rule again to obtain $\vdash^n x \in a, x : A, \Gamma \Rightarrow \Delta$. $\square$

Lemma 3.4.5. The rules of contraction are height-preserving admissible in **G3CL***.

$$\frac{\mathcal{F}, \mathcal{F}, \Gamma \Rightarrow \Delta}{\mathcal{F}, \Gamma \Rightarrow \Delta}\ \mathsf{Ctr_L} \qquad \frac{\Gamma \Rightarrow \Delta, \mathcal{F}, \mathcal{F}}{\Gamma \Rightarrow \Delta, \mathcal{F}}\ \mathsf{Ctr_R}$$

Proof. The proof is by simultaneous induction on the height of the derivation. Suppose $\vdash^n \mathcal{F}, \mathcal{F}, \Gamma \Rightarrow \Delta$; then we have to show that $\vdash^n \mathcal{F}, \Gamma \Rightarrow \Delta$, for $\mathcal{F}$ any formula of the language. If $n = 0$, the sequent $\mathcal{F}, \mathcal{F}, \Gamma \Rightarrow \Delta$ is an initial sequent; thus, also $\mathcal{F}, \Gamma \Rightarrow \Delta$ is an initial sequent. If $n > 0$, we look at the last rule (R) applied. If $\mathcal{F}$ is not principal in the rule, it suffices to apply the inductive hypothesis to the premiss of (R) and then (R). If $\mathcal{F}$ is the principal formula of (R) (or: was introduced by R) we distinguish two subcases.

a) The last rule is one in which the principal formulas appear also in the premiss (such as $\mathsf{L} \Vdash^{\forall}$, $\mathsf{R} \Vdash^{\exists}$, $\mathsf{L} >$, $\mathsf{R}|$, or one of the rules for the extensions). In these cases we apply the inductive hypothesis to the premiss(es) and then the rule. For instance, if the last rule use to derive the premiss of contraction is $\mathsf{R}|$ we have:

$$\frac{\begin{array}{l} a \in N(x), \Gamma \Rightarrow \Delta, x \Vdash_a A|B, x \Vdash_a A|B, a \Vdash^{\exists} A \\ \quad\vdots \qquad\qquad a \in N(x), \Gamma \Rightarrow \Delta, x \Vdash_a A|B, x \Vdash_a A|B, a \Vdash^{\forall} A \to B \end{array}}{a \in N(x), \Gamma \Rightarrow \Delta, x \Vdash_a A|B, x \Vdash_a A|B}\ \mathsf{R}|$$

By inductive hypothesis applied to the premiss, of smaller height, we get $a \in N(x), \Gamma \Rightarrow \Delta, x \Vdash_a A|B, a \Vdash^{\exists} A$ and $a \in N(x), \Gamma \Rightarrow \Delta, x \Vdash_a A|B, a \Vdash^{\forall} A \to B$ and thus by a step of $\mathsf{R}|$ we obtain $a \in N(x), \Gamma \Rightarrow \Delta, x \Vdash_a A|B$, with the same derivation height of the contraction premiss.

b) The last rule is one in which the active formulas are proper subformulas of the principal formula and possibly relational atoms (such as the rules for $\wedge$, $\vee$, $\to$, $\mathsf{R} \Vdash^{\forall}$, $\mathsf{L} \Vdash^{\exists}$, $\mathsf{R} >$, $\mathsf{L}|$). In all such cases, we apply invertibility to the premiss(es) of the rule so that we have a duplication of formulas at a smaller derivation height, then apply the inductive hypothesis (as many times as needed) and finally the rule in question. For instance, suppose the last rule applied is $\mathsf{R} >$, to a derivation of n height. We have:

$$\frac{a \in N(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, x : A > B}{\Gamma \Rightarrow \Delta, x : A > B, x : A > B}\ \mathsf{R} >$$

Using height-preserving invertibility of $\mathsf{R} >$ we obtain from the premiss a derivation of height $n - 1$ of

$$a \in N(x), a \in N(x), a \Vdash^{\exists} A, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, x \Vdash_a A|B.$$

By inductive hypothesis we get a derivation of the height $n-1$ of $a \in N(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B$ and application of $\mathsf{R}>$ gives a derivation of height n of $\Gamma \Rightarrow \Delta, x : A > B$. □

Theorem 3.4.6 (Cut-admissibility)**.** The rule of cut is admissible in **G3CL***.

$$\dfrac{\Gamma \Rightarrow \Delta, \mathcal{F} \qquad \mathcal{F}, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

Proof. The proof is by primary induction on the weight of the cut formula and on secondary induction on the sum of heights of the derivations of the premisses of cut[8]. We distinguish cases according to the rules applied to derive the premisses:

a) At least one of the premisses of cut is an initial sequent;
b) The cut formula is not the principal formula in the derivation of at least one premiss;
c) The cut formula is the principal formula of both derivations of the premisses.

Case a) Suppose the leftmost premiss $\Gamma \Rightarrow \Delta, \mathcal{F}$ is an initial sequent. If $\mathcal{F} = x : p$ and $x : p$ occurs in Γ, the situation is the following:

$$\dfrac{\overset{(1)}{x : p, \Gamma \Rightarrow \Delta, x : p} \quad \overset{(2)}{x : p, \Gamma' \Rightarrow \Delta'}}{x : p, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

In this case, the conclusion of cut can be obtained by weakening from (2). If the sequent is an initial sequent in virtue of some formulas $x : P$ occurring in Γ and Δ, also sequent $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is an initial sequent. If the rightmost premiss is an initial sequent, the same holds. If $\mathcal{F} = x : \bot$, $x : \bot$ is the cut formula and the left premiss is not an instance of init, it has been derived by some rule R. If R is $\bot_{\mathsf{L}}$, then $x : \bot$ occurs in the conclusion of cut. Since it occurs also in the conclusion of $\bot_{\mathsf{L}}$, the occurrence of cut can be eliminated. If the left premiss has been derived by a rule R different from $\bot_{\mathsf{L}}$, we permute the cut upwards on the left premiss.

Case b) Suppose the cut formula is not principal in the last rule applied to the left premiss of cut.

$$\dfrac{\dfrac{\overset{(1)}{\Gamma^* \Rightarrow \Delta^*, \mathcal{F}}}{\Gamma \Rightarrow \Delta, \mathcal{F}}\ R \qquad \overset{(2)}{\mathcal{F}, \Gamma' \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

The cut is eliminated as follows, where the application of the inductive hypothesis is justified by a smaller derivation height.

$$\dfrac{\dfrac{\overset{(1)}{\Gamma^* \Rightarrow \Delta^*, \mathcal{F}} \quad \overset{(2)}{\mathcal{F}, \Gamma' \Rightarrow \Delta'}}{\Gamma^*, \Gamma' \Rightarrow \Delta^*, \Delta'}\ \mathsf{cut}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ R$$

Some special care is needed in case of Repl_1 and Repl_2. Suppose the right premises of cut is derived by Repl_1:

$$\dfrac{\overset{(1)}{\Gamma \Rightarrow \Delta, x : p} \qquad \dfrac{\overset{(2)}{y \in \{x\}, x : p, y : p, \Gamma' \Rightarrow \Delta'}}{y \in \{x\}, x : p, \Gamma' \Rightarrow \Delta'}\ \mathsf{Repl}_1}{y \in \{x\}, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

[8] Refer to [74] for the general methodology of proving cut-admissibility in labelled systems.

Since p is an atomic formula, there are three possible cases to consider, according to how (1) has been derived. If (1) is an initial sequent and $x : p$ occurs in Γ, the conclusion can be obtained applying multiple weakening to the right premiss of cut.

$$\dfrac{y \in \{x\}, x : p, \Gamma' \Rightarrow \Delta'}{y \in \{x\}, x : p, \Gamma^*, \Gamma', \Rightarrow \Delta, \Delta'}\ \mathsf{Wk_L}$$

The same holds if (1) is an initial sequent or $x : \bot$ occurs in Γ. If (1) is not an initial sequent, it must have been derived by some rule, and $x : p$ cannot be the principal formula of the rule, since p is atomic. Thus, by inductive hypothesis we apply cut to the premiss of (1) and the right premiss of the original cut occurrence, which have a smaller sum of heights. The case in which the right premiss of cut is derived by $\mathsf{Repl_2}$ is symmetrical.

Case c) Propositional cases are immediate; refer to [74, Theorem 3.2.3] for details. The rules for extensions of **G3CL** are not involved in this case: relational atoms occur only in the antecedent of sequents, and the rules do not modify the rest of the sequent. Similarly, the cases of $\mathsf{Repl_1}$ and $\mathsf{Repl_2}$ do not obtain, since the rules adds formulas only in the antecedent.

$\mathcal{F} = a \Vdash^{\exists} A$:

$$\dfrac{\dfrac{\begin{matrix}(1)\\ x \in a, \Gamma \Rightarrow \Delta, x : A\end{matrix}}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{R}\Vdash^{\exists} \quad \dfrac{\begin{matrix}(2)\\ y \in a, y : A, \Gamma' \Rightarrow \Delta'\end{matrix}}{a \Vdash^{\exists} A, \Gamma' \Rightarrow \Delta'}\ \mathsf{L}\Vdash^{\exists}}{x \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

The cut is eliminated by means of the following permutation, where the upper application of cut is performed on formulas of smaller weight, and the lower is performed on formulas of smaller height.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}(1)\\ x \in a, \Gamma \Rightarrow \Delta, x : A\end{matrix} \quad \begin{matrix}(2)[y/x]\\ x \in a, x : A, \Gamma' \Rightarrow \Delta'\end{matrix}}{x \in a, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', a \Vdash^{\exists} A}\ \mathsf{cut} \quad a \Vdash^{\exists} A, \Gamma' \Rightarrow \Delta'}{x \in a, x \in a, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}\ \mathsf{cut}}{x \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{Ctr}$$

$\mathcal{F} = x \Vdash_a A|B$:

$$\dfrac{\dfrac{(1) \qquad (2)}{c \in N(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}\ \mathsf{R|} \quad \dfrac{(3)}{x \Vdash_a A|B, \Gamma' \Rightarrow \Delta'}\ \mathsf{L|}}{c \in N(x), c \subseteq a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

Where (1) $= c \in N(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, c \Vdash^{\exists} A$; (2) $= c \in N(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, c \Vdash^{\forall} A \to B$ and (3) $= d \in N(x), d \subseteq a, d \Vdash^{\exists} A, d \Vdash^{\forall} A \Rightarrow B, \Gamma' \Rightarrow \Delta'$. First, we apply cut on premisses of $\mathsf{R|}$. These applications of Cut have sum of height of the premisses smaller than the original occurrence of Cut:

$$\mathcal{D}_1 \quad = \quad \dfrac{(1) \qquad x \Vdash_a A|B, \Gamma' \Rightarrow \Delta'}{c \in N(x), c \subseteq a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c \Vdash^{\exists} A}\ \mathsf{cut}$$

$$\mathcal{D}_2 \quad = \quad \dfrac{(2) \qquad x \Vdash_a A|B, \Gamma' \Rightarrow \Delta'}{c \in N(x), c \subseteq a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c \Vdash^{\forall} A \to B}\ \mathsf{cut}$$

Then we apply two further cuts, on formulas of a smaller derivation weight.

$$
\dfrac{\mathcal{D}_2 \qquad \dfrac{\mathcal{D}_1 \qquad \overset{(3)[d/c]}{c \in N(x), c \subseteq a, c \Vdash^{\exists} A, c \Vdash^{\forall} A \Rightarrow B, \Gamma' \Rightarrow \Delta'}}{c \in N(x)^2, c \subseteq a^2, \Gamma, \Gamma'^2 \Rightarrow \Delta, \Delta'^2, c \Vdash^{\forall} A \rightarrow B}\,\mathsf{cut}}{\dfrac{c \in N(x)^3, c \subseteq a^3, \Gamma^2, \Gamma'^3 \Rightarrow \Delta^2, \Delta'^3}{c \in N(x), c \subseteq a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathsf{Ctr}}\,\mathsf{cut}
$$

$\mathcal{F} = x : A > B$:

$$
\dfrac{\dfrac{\overset{(1)}{b \in N(x), b \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}}{\Gamma \Rightarrow \Delta, x : A > B}\,\mathsf{R>} \qquad \dfrac{\overset{(2)}{.. \Rightarrow \Delta', a \Vdash^{\exists} A} \quad \overset{(3)}{x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma' \Rightarrow \Delta'}}{a \in N(x), x : A > B, \Gamma' \Rightarrow \Delta'}\,\mathsf{L>}}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathsf{cut}
$$

We first apply cut on the premisses of $\mathsf{L} >$. Both applications have a smaller sum of height of the premisses with respect to the original application of cut:

$$
\mathcal{D}_1 = \quad \dfrac{\Gamma \Rightarrow \Delta, x : A > B \quad \overset{(2)}{a \in N(x), x : A > B, \Gamma' \Rightarrow \Delta', a \Vdash^{\exists} A}}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathsf{cut}
$$

$$
\mathcal{D}_2 = \quad \dfrac{\Gamma \Rightarrow \Delta, x : A > B \quad \overset{(3)}{x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma' \Rightarrow \Delta'}}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathsf{cut}
$$

We combine the above with two occurrences of cut, on formulas of lesser weight than the original cut formula.

$$
\dfrac{\dfrac{\mathcal{D}_1 \qquad \overset{(1)[b/a]}{b \in N(x), b \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}}{a \in N(x)^2, \Gamma^2, \Gamma' \Rightarrow \Delta^2, \Delta', x \Vdash_a A|B}\,\mathsf{cut} \qquad \mathcal{D}_2}{\dfrac{a \in N(x)^3, \Gamma^3, \Gamma'^2 \Rightarrow \Delta^3, \Delta'^2}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathsf{Ctr}}\,\mathsf{cut}
$$

□

The axioms and inference rules of $\mathbb{PCL}$ and its extensions can be derived or proved to be admissible in the calculi **G3CL***. To prove admissibility of *modus ponens* we need admissibility of cut.

Theorem 3.4.7. If a formula A is valid in $\mathbb{PCL}$ or in one of its extensions, then there is a derivation of $\Rightarrow x : A$ in the calculus **G3CL*** for the corresponding logic.

Proof. Refer to Appendix 3.A at the end of this chapter. □

We conclude by proving admissibility of rules Repl_1 and Repl_2 in their generalized form. This lemma will be used in Section 3.6, to prove completeness of the calculi.

Lemma 3.4.8. Rules Repl_1 and Repl_2 generalized to all formulas of the language are admissible in **G3CL***.

Proof. Admissibility of the two rules is proven simultaneously, by induction on the weight of formulas. We only show the proof admissibility for Repl_1 (the other rule is symmetric). Since contraction and cut are admissible in **G3CL***(and their admissibility is independent of this result) it is sufficient to show that the sequent $y \in \{x\}, A(x) \Rightarrow A(y)$ is derivable. Once we have derivability of this sequent, the conclusion of Repl_1 can be derived from its premiss applying cut and contraction.

We proceed by induction on the weight of formula $A(x)$; there are several cases to consider.

1. $A(x) \equiv x : \mathcal{F}$, $A(y) \equiv y : \mathcal{F}$, where $\mathcal{F}$ is a propositional formula. We consider the case $A(x) \equiv x : B \to C$, $A(y) \equiv y : B \to C$.

$$\dfrac{\dfrac{\dfrac{\overset{\triangledown}{y \in \{x\}, x : B, y : B \Rightarrow y : C, x : B}}{y \in \{x\}, y : B \Rightarrow y : C, x : B}\,\mathsf{Repl}_2 \quad \dfrac{\overset{\triangledown}{y \in \{x\}, y : B, x : C, y : C \Rightarrow y : C, x : B}}{y \in \{x\}, y : B, x : C \Rightarrow y : C}\,\mathsf{Repl}_1}{y \in \{x\}, x : B \to C, y : B \Rightarrow y : C}\,\mathsf{L}\to}{y \in \{x\}, x : B \to C \Rightarrow y : B \to C}\,\mathsf{R}\to$$

In this case we need Repl_2, applied to formulas of smaller weight, and the two premisses are derivable by Lemma 3.4.2.

2. $A(x) \equiv x \Vdash_a B|C$, $A(y) \equiv y \Vdash_a B|C$.

$$\dfrac{\dfrac{\dfrac{(1) \qquad\qquad (2)}{c \in N(y), c \in N(x), c \subseteq a, c \Vdash^{\exists} B, c \Vdash^{\forall} B \to C, y \in \{x\} \Rightarrow y \Vdash_a B|C}\,\mathsf{R}|}{c \in N(x), c \subseteq a, c \Vdash^{\exists} B, c \Vdash^{\forall} B \to C, y \in \{x\} \Rightarrow y \Vdash_a B|C}\,\mathsf{Repl}_1}{y \in \{x\}, x \Vdash_a B|C \Rightarrow y \Vdash_a B|C}\,\mathsf{L}|$$

Where (1) $= c \in N(y), c \in N(x), c \subseteq a, c \Vdash^{\exists} B, c \Vdash^{\forall} B \to C, y \in \{x\} \Rightarrow y \Vdash_a B|C, c \Vdash^{\exists} A$ and (2) $= c \in N(y), c \in N(x), c \subseteq a, c \Vdash^{\exists} B, c \Vdash^{\forall} B \to C, y \in \{x\} \Rightarrow y \Vdash_a B|C, c \Vdash^{\forall} B \to C$. Rule Repl_1 is applied to the atomic formula $c \in N(x)$, which has smaller weight than $A(x)$. The lower premiss is derivable by Lemma 3.4.2, the upper one by steps of $\mathsf{L}\Vdash^{\exists}$, $\mathsf{L}\Vdash^{\forall}$, $\mathsf{L}\subseteq$, and Lemma 3.4.2.

3. $A(x) \equiv x : B > C$, $A(y) \equiv y : B > C$.

$$\dfrac{\dfrac{\dfrac{x \Vdash_a B|C, a \in N(x), a \in N(y), a \Vdash^{\exists} B, y \in \{x\}, x : B > C \Rightarrow y \Vdash_a B|C}{a \in N(x), a \in N(y), a \Vdash^{\exists} B, y \in \{x\}, x : B > C \Rightarrow y \Vdash_a B|C}\,L>}{a \in N(y), a \Vdash^{\exists} B, y \in \{x\}, x : B > C \Rightarrow y \Vdash_a B|C}\,Repl_2}{y \in \{x\}, x : B > C \Rightarrow y : B > C}\,R>$$

In this case we apply Repl_2 to formula $a \in N(y)$, of smaller weight. The leftmost premiss is the sequent $a \in N(x), a \in N(y), a \Vdash^{\exists} B, y \in \{x\}, x : B > C \Rightarrow y \Vdash_a B|C, a \Vdash^{\exists} A$, derivable by Case 1. $\Box$

With Gentzen calculi for classical propositional logic, admissibility of cut ensures the subformula property and its immediate consequences, such as consistency of the calculus, which follows from the underivability of the empty sequent. For the labelled calculi **G3CL*** the formulation of the subformula property needs to be modified. Taken literally, the subformula property would impose that any sequent occurring in a derivation of a given sequent $\Gamma \Rightarrow \Delta$ contains only formulas which are subformulas of the formulas in $\Gamma \Rightarrow \Delta$. However, the decomposition of a formula such as $A > B$ may introduce a formula $A|B$, and this latter might introduce $A \to B$. Neither $A|B$ nor $A \to B$ are, strictly speaking, subformulas of $A > B$ and $A|B$ respectively. However, according to Definition 3.2.1, these formulas have a smaller weight with respect to conditional formulas: in this sense, they are less complex, and can be considered as "generated" from a conditional formula. Then, we have to consider the labels: There are rules that may introduce arbitrary labels when read bottom-up. However, it can be proved that such rules can be restricted to a rule that operates on labels already in the conclusion thereby justifying the fact that in a **G3CDL** derivation all labels are either eigenvariables in rules with freshness condition, or labels already in the conclusion.

Thus, we can conclude that **G3CL*** is analytic and has the subformula property, once the notion of subformula has been extended to account for the cases described above.

3.5 Termination

The calculi **G3CL*** are not terminating by themselves: several cases of looping might occur in root-first proof search. However, it is possible to define a strategy under which proof search terminates. This is done by introducing the notion of *saturated sequent*, that is, a sequent to which all the applicable rules have been applied. Then, the *proof search strategy* defines an order in the application of the rules and prevents the application of any rules to a saturated sequent.

In this section we specify the proof search strategy and prove termination for calculi **G3CL**, $\mathbf{G3CL^N}$, $\mathbf{G3CL^T}$, $\mathbf{G3CL^W}$ and $\mathbf{G3CL^C}$. Termination for **G3CL** was already proved in [75]; we uniformly extend the proof to the aforementioned systems[9]. Termination for calculi with nesting and local absoluteness will be dealt with in the following sections. We do not prove termination for systems with local uniformity: the proof would require a more refined strategy, to deal with loops generated by the rules for uniformity. Moreover, logics with uniformity are known to be of the highest complexity in the family of Lewis' logics and the labelled calculus is not optimal - thus, the bound on the maximal number of labelled formulas that can be generated would be even higher than the (high) bound for logics without uniformity [28].

We start by reporting some examples of loops which might occur in naive root-first proof search in **G3CL** and $\mathbf{G3CL^N}$.

Example 3.5.1. Loop generated by repeated applications of rule $\mathsf{L} \Vdash^{\forall}$ to the same formula:

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{x \in a, x : A, x : C, x : C, a \Vdash^{\forall} C, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Vdash^{\forall}}{x \in a, x : A, x : C, a \Vdash^{\forall} C, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Vdash^{\forall}}{x \in a, x : A, a \Vdash^{\forall} C, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Vdash^{\forall}}{a \Vdash^{\exists} A, a \Vdash^{\forall} C, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Vdash^{\forall}$$

By height-preserving admissibility of contraction, we can contract the duplicated formulas in the upper sequent into one formula - and transform the derivation into a derivation in which no duplicated formulas occur. This and similar cases of loop can be resolved by imposing restrictions on bottom-up application of the rules.

[9] We actually prove completeness for a slightly different - and equivalent - version of the calculus. The reason for this will be clear in a few pages.

Remark 3.5.1. Loop generated by applications of rule N followed by rule $\mathsf{0}$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ z \in b, b \in N(x), y \in a, a \in N(x), \Gamma \Rightarrow \Delta\end{array}}{b \in N(x), y \in a, a \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{0}}{y \in a, a \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{N}}{a \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{0}}{\Gamma \Rightarrow \Delta}\ \mathsf{N}
$$

The side conditions of rule $\mathsf{0}$ block the loop: rule $\mathsf{0}$ is not applicable in case there is already a $w \in a$ belonging to the derivation, and is applicable only if there is at least one formula $a \Vdash^{\exists} A$ or $a \Vdash^{\forall} A$ occurring in $\Gamma \cup \Delta$. This ensures that the rule is not applied to "meaningless" formulas $b \in N(y)$ which could be introduced by N but are not subsequently used in the derivation.

Example 3.5.2. Loop generated by repeated applications of $\mathsf{L}>$ and $\mathsf{L}|$ to the same conditional formula (we show only the left premiss of $\mathsf{L}>$):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ d \in N(x), d \subseteq c, d \Vdash^{\exists} P, d \Vdash^{\forall} P \to Q \ldots x : P > Q \ \Rightarrow x \Vdash_a P|R\end{array}}{\Vdash_a P|Q, c \in N(x), c \subseteq a, c \Vdash^{\exists} P, c \Vdash^{\forall} P \to Q, a \in N(x), a \Vdash^{\exists} P, x : P > Q \Rightarrow x \Vdash_a P|R}\ \mathsf{L}\subseteq}{c \in N(x), c \subseteq a, c \Vdash^{\exists} P, c \Vdash^{\forall} P \to Q, a \in N(x), a \Vdash^{\exists} P, x : P > Q \Rightarrow x \Vdash_a P|R}\ \mathsf{L}>}{\Vdash_a P|R, a \in N(x), a \Vdash^{\exists} P, x : P > Q \Rightarrow x \Vdash_a P|R}\ \mathsf{L}\subseteq}{a \in N(x), a \Vdash^{\exists} P, x : P > Q \Rightarrow x \Vdash_a P|R}\ \mathsf{L}>}{x : P > Q \Rightarrow x : P > R}\ \mathsf{R}>
$$

This loop is generated by the interplay between world and neighbourhood labels; it can be blocked by specifying saturation conditions on the rules (refer to Remark 3.5.2).

Example 3.5.3. Loop generated by repeated applications of rules $\mathsf{L}>$ and $\mathsf{L}|$ to two different conditional formulas.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ d \in N(x), d \subseteq b, d \Vdash^{\exists} A, d \Vdash^{\forall} A \to B, c \subseteq b, \ldots x : A > B, x : C > D \Rightarrow \Delta\end{array}}{x \Vdash_b C|D, c \in N(x), c \subseteq b, c \Vdash^{\exists} C, c \Vdash^{\forall} C \to D, b \in N(x), b \subseteq a, b \Vdash^{\exists} A, b \Vdash^{\forall} A \Rightarrow B, x : A > B, x : C > D \Rightarrow \Delta}\ \mathsf{L}|}{c \in N(x), c \subseteq b, c \Vdash^{\exists} C, c \Vdash^{\forall} C \to D, b \in N(x), b \subseteq a, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, x : A > B, x : C > D \Rightarrow \Delta}\ \mathsf{L}>}{x \Vdash_b C|D, b \in N(x), b \subseteq a, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, x : A > B, x : C > D \Rightarrow \Delta}\ \mathsf{L}|}{b \in N(x), b \subseteq a, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, x : A > B, x : C > D \Rightarrow \Delta}\ \mathsf{L}>}{x \Vdash_a A|B, x : A > B, x : C > D \Rightarrow \Delta}\ \mathsf{L}|}{a \in N(x), x : A > B, x : C > D \Rightarrow \Delta}\ \mathsf{L}>
$$

Differently from the former case, the saturation conditions on the rules are not enough to stop this case of loop. It is thus necessary to slightly modify the rules of **G3CL** [75]. We shall see in the proof of Lemma 3.5.4 how the modified version of the calculus prevents the loop.

Definition 3.5.1. We modify rule $\mathsf{L}>$ of **G3CL** into rule $\mathsf{L}>'$ and introduce rule $\mathsf{Mon}\forall$ in the sequent calculus.

$$
\dfrac{a \in N(x), x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \qquad a \Vdash^{\exists} A, x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), x : A > B, \Gamma \Rightarrow \Delta}\ \mathsf{L}>'
$$

$$
\dfrac{b \subseteq a, b \Vdash^{\forall} A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{b \subseteq a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ \mathsf{Mon}\forall
$$

With respect to rule $\mathsf{L}>$, in rule $\mathsf{L}>'$ the formula expressing the existential forcing of the antecedent of the conditional (here $a \Vdash^{\exists} A$) is added to the rightmost premiss.

Lemma 3.5.1. Rule $\mathsf{Mon}\forall$ is admissible in **G3CL**.

Proof. Immediate, by induction on the height of derivations. □

Lemma 3.5.2. Rules $\mathsf{L}>$ and $\mathsf{L}>'$ are equivalent.

Proof. To prove that $\mathsf{L}>'$ implies $\mathsf{L}>$ apply weakening to the right premiss of $\mathsf{L}>$ and then apply $\mathsf{L}>'$ to obtain the conclusion of $\mathsf{L}>$. To prove that $\mathsf{L}>$ implies $\mathsf{L}>'$ we need cut, applied to the premisses of $\mathsf{L}>'$. Let (1) be leftmost premiss of $\mathsf{L}>'$, i.e., sequent $a \in N(x), x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$, and (2) the rightmost premiss, i.e., sequent $a \Vdash^{\exists} A, x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma \Rightarrow \Delta$. The conclusion of $\mathsf{L}>'$ is derived as follows:

$$\dfrac{(1) \qquad \dfrac{\dfrac{(1)}{x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\,\mathsf{Wk_L} \qquad (2)}{x \Vdash_a A|B, a \in N(x), x : A > B, \Gamma \Rightarrow \Delta}\,\mathsf{cut}}{a \in N(x), x : A > B, \Gamma \Rightarrow \Delta}\,\mathsf{L}>$$

□

Next we define the notion of saturated sequent and a proof search strategy ensuring termination for the calculi.

Definition 3.5.2. Given a derivation in **G3CL**, $\mathbf{G3CL^N}$, $\mathbf{G3CL^T}$, $\mathbf{G3CL^W}$ or $\mathbf{G3CL^C}$, let $\mathcal{B} = S_0, S_1, \dots$ be a derivation branch, with S_i sequent $\Gamma_i \Rightarrow \Delta_i$, for $i = 1, 2, \dots$ and S_0 sequent $\Rightarrow x : A_0$. Let $\downarrow \Gamma_k / \downarrow \Delta_k$ denote the union of the antecedents/succedents occurring in the branch from S_0 up to S_k.

We say that a sequent $\Gamma \Rightarrow \Delta$ *satisfies the saturation condition w.r.t. a rule* R if, whenever $\Gamma \Rightarrow \Delta$ contains the principal formulas in the conclusion of R, then it also contains the formulas introduced by *one* of the premisses of R. The saturation conditions are detailed below. Furthermore, $\Gamma \Rightarrow \Delta$ is *saturated* if there is no formula $x : p$ occurring in $\Gamma \cap \Delta$, there is no formula $x : \bot$ occurring in Γ and $\Gamma \Rightarrow \Delta$ satisfies *all* saturation conditions listed below.

- ($\mathsf{L}\wedge$) If $x : A \wedge B$ occurs in $\downarrow \Gamma$, then $x : A$ and $x : B$ occur in $\downarrow \Gamma$;
- ($\mathsf{R}\wedge$) If $x : A \wedge B$ occurs in $\downarrow \Delta$, then either $x : A$ or $x : B$ occur in $\downarrow \Delta$ occurs in $\downarrow \Delta$;
- ($\mathsf{L}\vee$) If $x : A \vee B$ occurs in $\downarrow \Gamma$, then either $x : A$ or $x : B$ occur in $\downarrow \Gamma$;
- ($\mathsf{R}\vee$) If $x : A \vee B$ occurs in $\downarrow \Delta$, then $x : A$ and $x : B$ occur in $\downarrow \Delta$;
- ($\mathsf{L}\rightarrow$) If $x : A \rightarrow B$ occurs in $\downarrow \Gamma$, then either $x : B$ occurs in $\downarrow \Gamma$ or $x : A$ occurs in $\downarrow \Delta$;
- ($\mathsf{R}\rightarrow$) If $x : A \rightarrow B$ occurs in Δ, then $x : A$ occurs in $\downarrow \Gamma$ and $x : B$ occurs in $\downarrow \Delta$;
- (Ref) If a occurs in $\Gamma \cup \Delta$, Δ then $a \subseteq a$ is in Γ[10];
- (Tr) If $a \subseteq b$ and $b \subseteq c$ are in Γ, then $a \subseteq c$ is in Γ;
- ($\mathsf{L}\subseteq$) If $x \in a$ and $a \subseteq b$ are in Γ, then $x \in b$ is in Γ;
- ($\mathsf{L}\Vdash^{\forall}$) If $x \in a$ and $a \Vdash^{\forall} A$ are in Γ, then $x : A$ is in $\downarrow \Gamma$;
- ($\mathsf{R}\Vdash^{\forall}$) If $a \Vdash^{\forall} A$ is in $\downarrow \Delta$, then for some x there is $x \in a$ in Γ and $x : A$ in $\downarrow \Delta$;
- ($\mathsf{L}\Vdash^{\exists}$) If $a \Vdash^{\exists} A$ is in $\downarrow \Gamma$, then for some x there is $x \in a$ in Γ and $x : A$ is in $\downarrow \Gamma$;
- ($\mathsf{R}\Vdash^{\exists}$) If $x \in a$ is in Γ and $a \Vdash^{\exists} A$ is in Δ, then $x : A$ is in $\downarrow \Delta$;
- ($\mathsf{Mon}\forall$) If $b \subseteq a$ and $a \Vdash^{\forall} A$ are in Γ, then $b \Vdash^{\forall} A$ is in Γ;

[10] In this case we do not need to specify $\downarrow \Gamma$: once introduced, world and neighbourhood labels do not disappear (bottom-up) from the derivation, and can be found in the upper sequent. The same holds for relational atoms.

($\mathsf{R}>$) If $x : A > B$ is in $\downarrow \Delta$, then for some a, $a \in N(x)$ is in Γ, $a \Vdash^{\exists} A$ is in $\downarrow \Gamma$ and $x \Vdash_a B|A$ is in Δ;

($\mathsf{L}>'$) If $a \in N(x)$ and $x : A > B$ are in Γ, then either $a \Vdash^{\exists} A$ is in $\downarrow \Delta$ or $a \Vdash^{\exists} A$ and $x \Vdash_a B|A$ are in $\downarrow \Gamma$;

($\mathsf{R}|$) If $c \in N(x)$ and $c \subseteq a$ are in Γ and $x \Vdash_a B|A$ is in Δ, then either $c \Vdash^{\exists} A$ is in Δ or $c \Vdash^{\forall} A \rightarrow B$ is in $\downarrow \Delta$;

($\mathsf{L}|$) If $x \Vdash_a B|A$ is in $\downarrow \Gamma$, then for some c, $c \in N(x)$ and $c \subseteq a$ are in Γ, $c \Vdash^{\exists} A$ is in $\downarrow \Gamma$ and $c \Vdash^{\forall} A \rightarrow B$ is in Γ.

($\mathsf{0}$) If $a \in N(x)$ occurs in Γ and some $a \Vdash^{\exists} A$ or $a \Vdash^{\forall} A$ occurs in $\downarrow \Gamma \cup \downarrow \Delta$ then $y \in a$ occurs in Γ, for some a;

(N) If x occurs in $\downarrow \Gamma \cup \downarrow \Delta$ then for some a, $a \in N(x)$ occurs in Γ;

(T) If x occurs in $\downarrow \Gamma \cup \downarrow \Delta$, there is an a such that $a \in N(x)$ and $x \in a$ are in Γ;

(W) If $a \in N(x)$ occurs in Γ then $x \in a$ occurs in Γ;

(C) If $a \in N(x)$ occurs in Γ, both $\{x\} \in N(x)$ and $\{x\} \subseteq a$ occur in Γ;

(Single) If $\{x\} \in N(x)$ occurs in Γ, then $x \in \{x\}$ occurs in Γ;

(Repl_1) If $y \in \{x\}$ occurs in Γ, and if some formula $At(x)$ occurs in Γ, then $At(y)$ occurs in Γ;

(Repl_2) If $y \in \{x\}$ occurs in Γ, and if some formula $At(y)$ occurs in Γ, then $At(x)$ occurs in Γ;

Definition 3.5.3. When constructing root-first a derivation tree for a sequent $\Rightarrow x_0 : A$, apply the following *proof search strategy*:

(i) Rule $\mathsf{R}>$ is applied before $\mathsf{L}>$;
(ii) Rule $\mathsf{0}$ is applied after all the other rules.
(iii) Rule T is applied as the first one to each world label x.
(iv) Rules which do not introduce a new label (static rules) are applied *before* the rules which do introduce new labels (dynamic rules), with the exception of $\mathsf{0}$ and T;
(v) A rule R cannot be applied to $\Gamma_i \Rightarrow \Delta_i$ if the sequent satisfies the saturation condition associated to (R).

Remark 3.5.2. The case of loop from Example 3.5.2 is stopped with the saturation condition for $\mathsf{L}|$. This condition blocks the application of the uppermost occurrence of $\mathsf{L}|$: if $x \Vdash_a A|B$ occurs in $\downarrow \Gamma$, then $c \subseteq b$, $c \Vdash^{\exists} A$ and $c \Vdash^{\forall} A \rightarrow B$ are already in $\downarrow \Gamma$, for a label c that is b itself. But the saturation condition is not enough to block the loop of of Example 3.5.3. Application of rule $\mathsf{L}|$ to formula $\Vdash_b C|D$ cannot be stopped: in the branch there is no $d \subseteq b$ such that $b \Vdash^{\exists} C$ and $b \Vdash^{\forall} C \rightarrow D$.

In order to show that the calculi are terminating under the proof-search strategy we reason as follows. We have to show that every branch of a derivation tree whose root is sequent $\Rightarrow x : A_0$ is finite. Since labels can be attached only to subformulas of the initial formula A_0, and since these subformulas are finitely many, it is sufficient to prove that that the labels for worlds and for neighbourhoods occurring in the derivation are finitely many. In order to prove this fact, we construct an *acyclic graph* with all the labels which may occur in a branch of the derivation (Definition 3.5.4). Then, to prove that the graph is finite, we need to prove the following facts:

1) Every node of the graph has a finite number of immediate successors (Lemma 3.5.4);
2) Every branch of the graph has a finite length (Lemma 3.5.5).

Definition 3.5.4. Given a derivation branch as in Definition 3.5.2, let a, b be neighbourhood labels and x, y world labels occurring in $\downarrow \Gamma_k$. We define:

- $k(x) = min\{t \mid x \ occurs \ in \ \Gamma_t\}$;

- $k(a) = min\{t \mid a \ occurs \ in \ \Gamma_t\}$;
- $x \rightarrow_g a$, "x generates a" if for some $t \leqslant k$ and $k(a) = t$, $a \in N(x)$ occurs in Γ_t;
- $b \rightarrow_g y$, "b generates y" if for some $t \leqslant k$ and $k(y) = t$, $y \in b$ occurs in Γ_t;
- $x \overset{w}{\rightarrow} y$ if for some a, $x \rightarrow_g a$ and $a \rightarrow_g y$ and $x \neq y$.

Given a branch $\mathcal{B}$ as in Definition 3.5.2 and a world label x, we define $\mathsf{Neigh}(x) = \{a \mid x \rightarrow_g a\}$ as the set of neighbourhood labels generated by x, and $\mathsf{World}(x) = \{y \mid x \overset{w}{\rightarrow} y\}$ as the set of world labels generated by x.

Intuitively, the relation $x \rightarrow_g a$ holds between x and a if $a \in N(x)$ is introduced at some stage in the derivation (thus, with an application of $\mathsf{R}>$ or $\mathsf{L}|$); similarly, the relation $b \rightarrow_g y$ holds between b and y if $y \in b$ is introduced in the derivation (thus, applying either $\mathsf{R} \Vdash^{\forall}$ or $\mathsf{L} \Vdash^{\exists}$). Observe that neighbourhood labels are not explicitly considered in the tree, which is generated by the relation $\overset{w}{\rightarrow}$ between world labels. However, to generate a new world label one needs first to generate a neighbourhood label; thus, also neighbourhood labels are "counted" in this definition.

Definition 3.5.5. The *conditional degree* of a formula A corresponds to the level of nesting of the conditional operator in A and is defined as follows:

- $d(p) = d(\bot) = 0$ for p atomic;
- $d(C \circ D) = max(d(C), d(D))$ for $\circ \in \{\wedge, \vee, \rightarrow\}$;
- $d(C > D) = max(d(C), d(D)) + 1$.

Lemma 3.5.3. Given a derivation branch as in Definition 3.5.2, the following hold:

(a) The relation $\overset{w}{\rightarrow}$ does not contain any cycles and forms a graph displaying at the root the world label x_0;
(b) All labels occurring in a derivation branch also occur in the associated graph; that is, letting $x \overset{w*}{\rightarrow} y$ be the transitive closure of $\overset{w}{\rightarrow}$, if u occurs in $\downarrow \Gamma_k$, then $x_0 \overset{w*}{\rightarrow} u$.

Proof. (a) immediately follows from the definition of relation $\overset{w}{\rightarrow}$ and from the sequent calculus rules. As for (b), it is easily proved by induction on $k(u) \leqslant k$. If $k(u) = 0$, then $u = x_0$ and (b) trivially holds. If $k(u) = t > 0$, u does not occur in Γ_{t-1} and u occurs in Γ_t. This means that there exist a v and a b such that $b \in N(v)$ occurs in Γ_{t-1} and $u \in b$ occurs in Γ_t; thus, $k(v) < k(u)$. By inductive hypothesis, $x_0 \overset{w*}{\rightarrow} v$; since $v \overset{w}{\rightarrow} u$, also $x_0 \overset{w*}{\rightarrow} u$ holds. □

Lemma 3.5.4. Given a branch $\mathcal{B}$ as in Definition 3.5.2 and a world label x, the size of $\mathsf{Neigh}(x)$ and $\mathsf{World}(x)$ is finite, more precisely: $|\mathsf{Neigh}(x)| = O(n \cdot n!)$ and $|\mathsf{World}(x)| = O(n^2 \cdot n!)$, where $n = d(A_0)$.

Proof. We first prove that $|\mathsf{Neigh}(x)| = O(n \cdot n!)$. By definition, a neighbourhood label a can be generated from a world label x if there exists a t such that a does not occur in Γ_s for all $s < t$ and $a \in N(x)$ occurs in Γ_t. This means that label a has been introduced in the derivation either by some rule for semantic conditions, by $\mathsf{R}>$ or by $\mathsf{L}|$.

In the case of rules for semantic conditions, a neighbourhood label can be introduced either by N, T or C (in this latter case, the neighbourhood will be $\{x\}$). However, for the saturation condition, these rules introduce at most one formula $a \in N(x)$ or $\{x\} \in N(x)$ for a certain label x. Thus, given a label x, we count 3 new neighbourhood labels which could be introduced by application of these rules[11].

[11] Observe that the rules of replacement do not introduce new neighbourhood labels; however, they might introduce new links between the nodes of the graph. Rules of replacement are the reason why the structure associated to the labels occurring in a derivation is an acyclic graph and not a tree.

In case formula $a \in N(x)$ is generated by $\mathsf{R} >$, the rule must have been applied to some formula $x : C > D$ occurring in $\downarrow \Delta_s$. The number of such formulas is finite: it is $O(n)$. Moreover, since formula $x : C > D$ disappears from the premiss(es), we have that at most $O(n)$ neighbourhood labels are introduced.

In case formula $a \in N(x)$ is generated by $\mathsf{L}|$, the situation is more complex. Rule $\mathsf{L}|$ may interact with rule $\mathsf{L} >'$, as shown to Example 3.5.2 and 3.5.3, generating a large number of new neighbourhood labels. We shall prove that, under the proof-search strategy, this number if *finite*.

As seen in Remark 3.5.2, the problematic case is the one in which $\mathsf{L} >'$ and $\mathsf{L}|$ are applied to multiple $>$-formulas in the succedent. Thus, suppose there are k formulas $x : E_1 > F_1, ..., x : E_k > F_k$ occurring in Γ_s (Figure 3.3 should help keeping track of the formulas). For a given $b \in N(x)$ occurring in Γ_s we introduce at most k formulas $x \Vdash_b E_1|F_1, ..., x \Vdash_b E_k|F_k$, by k applications of $\mathsf{L} >'$. Now, rule $\mathsf{L}|$ can be applied to each of these formulas generating k new neighbourhoods, $b_1 \in N(x), ..., b_k \in N(x)$, such that $b_1 \subseteq b, \ldots, b_k \subseteq b$. The rule also introduces in the derivation the formulas $b_1 \Vdash^{\exists} E_1, ..., b_k \Vdash^{\exists} E_k$ and $b_1 \Vdash^{\forall} E_1 \to F_1, ..., b_k \Vdash^{\forall} E_k \to F_k$. We now apply $\mathsf{L} >'$ to each $b_1, ..., b_k$, generating $k \cdot k$ new formulas $x \Vdash_{b_1} E_1|F_1, ..., x \Vdash_{b_1} E_k|F_k, ..., x \Vdash_{b_k} E_1|F_1, ..., x \Vdash_{b_k} E_k|F_k$. Following the strategy, we apply the static rule Ref k times, obtaining $b_1 \subseteq b_1, ..., b_k \subseteq b_k$, and then apply the dynamic rule $\mathsf{L}|$. In principle, we could generate $k \cdot k$ new neighbourhood labels; but the strategy prevents the application of $\mathsf{L}|$ to k formulas. In fact, for each $x \Vdash_{b_i} E_1|F_1, ..., x \Vdash_{b_i} E_k|F_k$, for $i \leqslant k$, $\mathsf{L}|$ cannot be applied to formula $x \Vdash_{b_i} E_i|F_i$. Suppose for instance that $i = 1$. In this case, we have k formulas $x \Vdash_{b_1} E_1|F_1, x \Vdash_{b_1} E_2|F_2, ..., x \Vdash_{b_1} E_k|F_k$ to which $\mathsf{L}|$ could be applied; but the saturation condition for $\mathsf{L}|$ blocks the application of the rule to formula $x \Vdash_{b_1} E_1|F_1$, since in the sequent there occur also formulas $b_1 \subseteq b_1$, $b_1 \Vdash^{\exists} E_1$ and $b_1 \Vdash^{\forall} E_1 \to F_1$, and the saturation condition associated to $\mathsf{L}|$ is satisfied. Thus, at this stage we generate only $k(k-1)$ new neighbourhood labels, namely $c^1_2, ..., c^1_k \subseteq b_1, ..., c^k_1, ..., c^k_h \subseteq b_k$, for $h = k-1$, along with the formulas $c^1_2 \Vdash^{\exists} E_2, ..., c^1_k \Vdash^{\exists} E_k, ..., c^k_1 \Vdash^{\exists} E_1, ..., c^k_h \Vdash^{\exists} E_h$ and $c^1_2 \Vdash^{\forall} E_2 \to F_2, ..., c^1_k \Vdash^{\forall} E_k \to F_k, ..., c^k_1 \Vdash^{\forall} E_1 \to F_1, ..., c^k_h \Vdash^{\forall} E_h \to F_h$.

We now apply rule $\mathsf{L} >'$ to all the new labels, generating $k \cdot k(k-1)$ new formulas $x \Vdash_{c^1_2} E_1|F_1, ..., x \Vdash_{c^1_2} E_k|F_k, \ldots, x \Vdash_{c^1_k} E_1|F_1, ..., x \Vdash_{c^1_k} \ldots, x \Vdash_{c^k_1} E_1|F_1, ..., x \Vdash_{c^1_k} E_k|F_k, \ldots x \Vdash_{c^k_h} E_1|F_1, ..., x \Vdash_{c^k_h}$. Now the modified rule $\mathsf{L} >'$ becomes relevant: the rule introduces also $c^1_2 \Vdash^{\exists} E_1, \ldots, c^1_2 \Vdash^{\exists} E_k, \ldots c^1_k \Vdash^{\exists} E_1, \ldots c^1_k \Vdash^{\exists} E_k \ldots c^k_1 \Vdash^{\exists} E_1, \ldots, c^k_1 \Vdash^{\exists} E_k, \ldots c^k_h \Vdash^{\exists} E_1, \ldots c^k_h \Vdash^{\exists} E_k$.

Now, before applying rule $\mathsf{L}|$ we apply two static rules: Ref and $\mathsf{Mon}\forall$. Thanks to these two rules, we are able to limit applications of $\mathsf{L}|$: the number of new neighbourhood labels produced is $k(k-1)(k-2)$. For instance, let us consider neighbourhood label c^1_2. The application of $\mathsf{L} >'$ introduces $x \Vdash_{c^1_2} E_1|F_1, x \Vdash_{c^1_2} E_2|F_2, ..., x \Vdash_{c^1_2} E_k|F_k$, and $c^1_2 \Vdash^{\exists} E_1, ..., c^1_2 \Vdash^{\exists} E_k$. As before, the application of $\mathsf{L}|$ to $x \Vdash_{c^1_2} E_2|F_2$ is blocked, since formulas $c^1_2 \Vdash^{\exists} E_2$ and $c^1_2 \Vdash^{\forall} E_2 \to F_2$ were introduced by the previous application of $\mathsf{L} >'$ and by Ref we have that $c^1_2 \subseteq c^1_2$. Then, the application of $\mathsf{Mon}\forall$ to $c^1_2 \subseteq b_1$ and $b_1 \Vdash^{\forall} E_1 \to F_1$ (which was introduced some steps ago) yields $c^1_2 \Vdash^{\forall} E_1 \to F_1$. This formula, along with $c^1_2 \Vdash^{\exists} E_1$ and $c^1_2 \subseteq c^1_2$, blocks the application of $\mathsf{L}|$ to $x \Vdash_{c^1_2} E_1|F_1$. Thus, for a label c^1_2, *two* application of $\mathsf{L}|$ are blocked. If we continue to apply $\mathsf{L} >'$, the next application of $\mathsf{L}|$ will yield $k(k-1)(k-2)(k-3)$ new neighbourhood labels, and so on.

Basically, applications of $\mathsf{L} >'$ to k $>$-formulas and h neighbourhood labels produce $k \cdot h$ new formulas of the kind $E|F$, while applications of $\mathsf{L}|$ to the same formulas produce a strictly decreasing number of neighbourhood labels: the first application produces k labels, the second $k(k-1)$, the second $k(k-1)(k-2)$ and so on.

$$\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots\\ c_2^1 \subseteq c_2^1, x \Vdash_{c_2^1} E_1|F_1, \quad x \Vdash_{c_2^1} E_2|F_2, c_2^1 \Vdash^{\forall} E_1 \to F_1, \dots, x \Vdash_{c_2^1} E_k|F_k & \dots & c_h^k \subseteq c_h^k, \dots x, c_h^k \Vdash^{\forall} E_k \to F_k, x \Vdash_{c_h^k} E_k|F_k\end{matrix}}{c_2^1 \subseteq c_2^1, x \Vdash_{c_2^1} E_1|F_1, c_2^1 \Vdash^{\exists} E_1, \dots, x \Vdash_{c_2^1} E_k|F_k, c_2^1 \Vdash^{\exists} E_k \quad \dots \quad c_h^k \subseteq c_h^k, x \Vdash_{c_h^k} E_1|F_1, c_h^k \Vdash^{\exists} E_1 \dots, x \Vdash_{c_h^k} E_k|F_k, c_h^k \Vdash^{\exists} E_k}\ \mathsf{Mon}\forall}{x \Vdash_{c_2^1} E_1|F_1, c_2^1 \Vdash^{\exists} E_1, \dots, x \Vdash_{c_2^1} E_k|F_k, c_2^1 \Vdash^{\exists} E_k \quad \dots \quad x \Vdash_{c_h^k} E_1|F_1, c_h^k \Vdash^{\exists} E_1, \dots, x \Vdash_{c_h^k} E_k|F_k, c_h^k \Vdash^{\exists} E_k}\ \mathsf{Ref}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \Vdash_{c_2^1} E_1|F_1, c_2^1 \Vdash^{\exists} E_1, \dots \quad \dots \quad \dots, x \Vdash_{c_h^k} E_k|F_k, c_h^k \Vdash^{\exists} E_k}{c_2^1 \subseteq b_1, c_2^1 \Vdash^{\exists} E_2, c_2^1 \Vdash^{\forall} E_2 \to F_2, \dots, c_k^1 \subseteq b_1, c_k^1 \Vdash^{\exists} E_k, c_k^1 \Vdash^{\forall} E_k \to F_k, \dots\ c_1^k \subseteq b_1, c_1^k \Vdash^{\exists} E_1, c_1^k \Vdash^{\forall} E_1 \to F_1, \dots, c_h^k \subseteq b_1, c_h^k \Vdash^{\exists} E_h, c_h^k \Vdash^{\forall} E_h \to F_h}\ \mathsf{L}>'}{b_1 \subseteq b_1, x \Vdash_{b_1} E_1|F_1, \quad x \Vdash_{b_1} E_2|F_2 \quad \dots \quad x \Vdash_{b_1} E_k|F_k \quad \dots \quad b_k \subseteq b_k, x \Vdash_{b_k} E_1|F_1 \quad \dots \quad x \Vdash_{b_k} E_h|F_h, \quad x \Vdash_{b_k} E_k|F_k}\ \mathsf{L}|}{x \Vdash_{b_1} E_1|F_1, x \Vdash_{b_1} E_1|F_1 \quad \dots \quad x \Vdash_{b_1} E_k|F_k \quad \dots \quad x \Vdash_{b_1} E_1|F_1, x \Vdash_{b_1} E_1|F_1 \quad \dots \quad x \Vdash_{b_1} E_k|F_k}\ \mathsf{Ref}}{b_1 \subseteq b, b_1 \in N(x), b_1 \Vdash^{\exists} E_1, b_1 \Vdash^{\forall} E_1 \to F_1 \quad \dots \quad b_k \subseteq b, b_k \in N(x), b_k \Vdash^{\exists} E_k, b_k \Vdash^{\forall} E_k \to F_k}\ \mathsf{L}>'}{x \Vdash_b E_1|F_1 \quad \dots \quad x \Vdash_b E_k|F_k}\ \mathsf{L}|}{b \in N(x), x : E_1 > F_1 \quad \dots \quad x : E_k > F_k}\ \mathsf{L}>'}{}$$

Not all formulas are shown; each rule corresponds to a multiple application of the rule.
In the upper application of of $\mathsf{L}|$, the rule is not applied on formulas $x \Vdash_{b_1} E_1|F_1$ and $x \Vdash_{b_k} E_k|F_k$. Let $h = k - 1$.

Figure 3.3: Termination under $\mathsf{L} >'$ and $L|$

Thus, since the rules for semantic conditions and $\mathsf{R}>$ are applied before $\mathsf{L}>$ and $\mathsf{L}|$, and they produce at most $O(n+3) = O(n)$ neighbourhood labels, the total number of neighbourhood labels that can be introduced from a world label is $\mathsf{Neigh}(x) = O(n \cdot n!)$.

We now prove $|\mathsf{World}(x)| = O(n^2 \cdot n!)$. By definition $y \in W(x)$ if and only if $x \xrightarrow{w} y$, i.e., if and only if for some neighbourhood label b it holds that $x \to_g b$ and $b \to_g y$. Let us consider $b \to_g y$. By definition, this means that there exists $t < k$ such that y does not occur in Γ_s for $s \leqslant t$ and $y \in b$ occurs in Γ_{t+1}. There are several ways in which a formula $y \in b$ can be introduced:

- $y \in b$ is introduced by $\mathsf{L} \Vdash^{\exists}$ applied to some $b \Vdash^{\exists} C$ occurring in Γ_t;
- $y \in b$ is introduced by $\mathsf{R} \Vdash^{\forall}$ applied to some $b \Vdash^{\forall} C$ occurring in Δ_t;
- $y \in b$ is introduced by $\mathsf{0}$ applied to some $b \in N(x)$, for some $b \Vdash^{\forall} C$ occurring in Γ_t or $b \Vdash^{\exists} C$ occurring in Δ_t.

In all cases the rule can be applied only once to each formula $b \Vdash^{\exists} C$, $b \Vdash^{\forall} C$ or $b \in N(x)$; moreover, $\mathsf{0}$ is not applied if any of the other two rules can be applied. The number of neighbourhood labels that can be introduced in the derivation linearly depends on the number of formulas $b \Vdash^{\exists} C$, $b \Vdash^{\forall} C$ that can be introduced.

Let us consider formulas $b \Vdash^{\exists} C$. There are two rules that introduce (bottom-up) this formulas:

a) Rule $\mathsf{R}>$ applied to a formula $x : D > C$ occurring in $\downarrow \Delta_k$. In this case, the rule can be applied only *once* to each formula $x : D > C$ that occurs in the consequent, generating exactly *one* new neighbourhood label. The number of formulas $x : D > C$ is $O(n)$, and thus the number of formulas $b \Vdash^{\exists} C$ that can be introduced is $O(n)$.
b) Rule $\mathsf{L}|$ applied to a formula $x \Vdash_b D|C$ occurring in $\downarrow \Gamma_k$. In turn, $x \Vdash_b D|C$ has been introduced by $\mathsf{L}>$ applied to a formula $x : D > C$ occurring in Γ_k. By the saturation condition rule $\mathsf{L}>$ can be applied only once to each neighbourhood $b \in N(x)$, producing one formula $x \Vdash_b D|C$ and one new world label. The number of formulas $x : D > C$ in $\downarrow \Gamma_k$ is $O(n)$; thus, the number of formulas $b \Vdash^{\exists} C$ is $O(n)$.

As for $b \Vdash^{\forall} C$, these formulas can be introduced bottom-up by application of rule $\mathsf{R}|$ to a formula $x \Vdash_c E|F$ occurring in $\downarrow \Gamma_k$, for $b \subseteq c$ also occurring in Γ_k. Again, formula $x \Vdash_c E|F$ is introduced by $\mathsf{R}>$ applied to some formula $x : E > F$ occurring in $\downarrow \Gamma_k$. Thus, the maximal number of formulas $b \Vdash^{\forall} C$ that can be introduced is bounded by $O(n)$.

Thus, each $b \in \mathsf{Neigh}(x)$ generates at most $O(n)$ neighbourhood labels. We have shown that $|\mathsf{Neigh}(x)| = O(n \cdot n!)$. The number of world labels that can be generated from a world label x is given by $|\mathsf{World}(x)| = O(n^2 \cdot n!)$. □

Lemma 3.5.5. Every branch in the graph generated by relation $\xrightarrow{w}$ chain is finite; more precisely, the length of any chain is $O(n)$, where $n = d(A_0)$.

Proof. For any world label x occurring in the branch, let

$$d(x) = max\{d(C) \mid x : C \in \downarrow \Gamma \cup \downarrow \Delta\}.$$

By induction on $d(x)$ we show that the length of an arbitrary chain from label x is bounded by the degree of the formula it labels. In symbols, the length of a chain beginning with x is $\leqslant d(x)$.

If $d(x) = 0$, the formulas labelled by x are all atomic formulas. Logical rules introduce new world labels in the derivation, but they cannot be applied, since all formulas are atomic. Rule $\mathsf{0}$ is the only non-logical rule that might introduce a new world label. However, this rule cannot be applied. By the side conditions imposed on the rule, $\mathsf{0}$ could be applied only to a sequent

$$a \in N(x), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$$

with no $w \in a$ occurring in Γ. Formula $a \Vdash^{\exists} A$ in the consequent might be introduced either by $\mathsf{L} >'$ (left premiss) applied to some $x : A > B$ occurring in Γ or by $\mathsf{R}|$ (left premiss) applied to some formula $x \Vdash_c A|B$ in $\downarrow \Delta$, in turn generated by $\mathsf{R} >$ applied to $x : A > B$. In both cases we would have that x labels formulas of degree greater than 0, against the hypothesis. Observe that rule $\mathsf{0}$ cannot be applied to a sequent $a \in N(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta$: the only way in which such a sequent can be generated is by application of rule $\mathsf{L}|$ to some formula $x \Vdash_c E|F$, for $A = E \to F$. However, $\mathsf{L}|$ also introduces in the antecedent rule $a \Vdash^{\exists} E$; by the proof-search strategy, rule $\mathsf{L} \Vdash^{\exists}$ has to be applied to this latter formula *before* rule $\mathsf{0}$, and afterwards rule $\mathsf{0}$ cannot be applied. Thus, for $d(x) = 0$, all chains starting from x have length 0.

If $d(x) > 0$, there must be some $x : C > D$ occurring in $\downarrow \Gamma \cup \downarrow \Delta$, and there will be at least one chain of length greater than 0 in the graph. Take one of these chains: it will contain a label y as immediate successor of x, and such that $a \in N(x)$ and $y \in a$ both occur in $\downarrow \Gamma \cup \downarrow \Delta$ (this means that label y has been introduced by some combinations of $\mathsf{L} >'$ or $\mathsf{R} >$, $\mathsf{L}|$ or $\mathsf{R}|$ and $\mathsf{0}$, $\mathsf{R} \Vdash^{\forall}$ or $\mathsf{L} \Vdash^{\exists}$).

Observe that, in this case, y can occur only as label of formulas which have a smaller degree than $C > D$. More precisely, for any $C > D$ such that $x : C > D$ occurs in $\downarrow \Gamma \cup \downarrow \Delta$, with $d(C > D) \leqslant d(x)$, and for all formulas $a \Vdash^{\forall} G$ or $a \Vdash^{\exists} G$ occurring in $\downarrow \Gamma \cup \downarrow \Delta$, with $G = C$ or $G = C \to D$, it holds that $d(G) < d(x)$. Thus, all formulas $y : E$ occurring in $\downarrow \Gamma \cup \downarrow \Delta$ may only be subformulas of a formula G such that $a \Vdash^{\exists} G$ or $a \Vdash^{\forall} G$, with $G = C$ or $G = C \to D$ are in $\downarrow \Gamma \cup \downarrow \Delta$ (i.e. the world label y have been generated from x by combinations of rules $\mathsf{R} >$ or $\mathsf{L}|$, $\mathsf{L} >'$ or $\mathsf{R}|$, and $\mathsf{L} \Vdash^{\exists}$ or $\mathsf{R} \Vdash^{\forall}$). The same holds in case the label y has been generated by $\mathsf{0}$: it might label only subformulas of some conditional formulas. It holds that $d(y) < d(x)$, and more precisely $d(x) = d(y)+1$. By inductive hypothesis all chains beginning with y have length $\leqslant d(y)$. The chain beginning with x has length $\leqslant (d(y) + 1) \leqslant ((d(x) - 1) + 1) = d(x)$. $\square$

Theorem 3.5.6 (Termination)**.** Proof search built in accordance with the strategy for a sequent of the form $\Rightarrow x_0 : A_0$ always comes to an end after a finite number of steps, and each sequent that occurs as a leaf of the derivation tree is either an initial sequent or a saturated sequent.

Proof. If the branch contains an initial sequent, this sequent is the last one and the branch is finite. If the branch does not contain an initial sequent, any label occurring in the consequent of the sequent also occurs in the antecedent (with the possible exception of the initial label $x_0 : A$). Build an acyclic graph with the labels according to the relation $\xrightarrow{w}$ as described above: since each node of the graph has a finite number of immediate successors (Lemma 3.5.4) since each chain of $\xrightarrow{w}$ is finite (Lemma 3.5.5), the graph is finite. As a consequence, also the proof search is finite, since there are only finitely many subformulas of the initial formula A and each can be labelled only by finitely many labels.

To prove the second part of the theorem, consider a branch $\Gamma_0 \Rightarrow \Delta_0, \ldots, \Gamma_n \Rightarrow \Delta_n$. As we have just proved, every branch of a derivation tree is finite. The leaf of the branch will be the sequent $\Gamma_n \Rightarrow \Delta_n$, and no rule is applicable to it; thus, trivially, the sequent is either an initial sequent or it is saturated. $\square$

3.6 Semantic completeness

Completeness for a system of sequent calculus can be proved either with respect to the axiom system or with respect to the class of models for the logic. Theorem 3.4.7, along with Theorem 3.4.6 of cut-admissibility, proved completeness of the calculi **G3CL*** with respect to the axioms of the corresponding logics. In this section we prove completeness of the calculi **G3CL**, **G3CL**$^{\mathbf{N}}$, **G3CL**$^{\mathbf{T}}$, **G3CL**$^{\mathbf{W}}$ and **G3CL**$^{\mathbf{C}}$ with respect to the corresponding classes of models. We refer to this theorem as *semantic completeness*. The proof consists in showing that, if a formula is valid in a model, then it is derivable in the sequent calculus. We prove the counterpositive statement: if a formula is not derivable in the sequent calculus, then we can construct a countermodel for it. The proof exploits in an essential way termination of the systems of sequent calculus. The proof for **G3CL** was given in [75]; we show how it can be extended to other systems.

Theorem 3.6.1. Let $\Gamma \Rightarrow \Delta$ be a saturated upper sequent in a derivation in **G3CL**, **G3CL**$^{\mathbf{N}}$, **G3CL**$^{\mathbf{T}}$, **G3CL**$^{\mathbf{W}}$ or **G3CL**$^{\mathbf{C}}$. There exists a *finite* countermodel $\mathcal{M}$ that satisfies all formulas in $\downarrow \Gamma$ and falsifies all formulas in $\downarrow \Delta$.

Proof. We start by considering derivations in **G3CL**, **G3CL**$^{\mathbf{N}}$, **G3CL**$^{\mathbf{T}}$ and **G3CL**$^{\mathbf{W}}$. The case of **G3CL**$^{\mathbf{C}}$ (preferential logic with centering) requires a different countermodel construction, to account for neighbourhood $\{x\}$.

Since $\Gamma \Rightarrow \Delta$ is saturated, it is the upper sequent of a branch $\mathcal{B}$. We construct a model $\mathcal{M}_{\mathcal{B}}$ that satisfies all formulas in $\downarrow \Gamma$ and falsifies all formulas in $\downarrow \Delta$. The countermodel contains the semantic informations encoded in the sequents of the derivation branch. Let

$$S_{\mathcal{B}} = \{x \mid x \in (\downarrow \Gamma \cup \downarrow \Delta)\} \qquad N_{\mathcal{B}} = \{a \mid a \in (\downarrow \Gamma \cup \downarrow \Delta)\}$$

Then, we associate to each $a \in N_{\mathcal{B}}$ a neighbourhood

$$\alpha_a = \{y \in S_{\mathcal{B}} \mid y \in a \textit{ belongs to } \Gamma\}$$

Thus, for each neighbourhood a, $\alpha_a \subseteq S_{\mathcal{B}}$. We construct the neighbourhood model $\mathcal{M}_{\mathcal{B}} = \langle W_{\mathcal{B}}, N_{\mathcal{B}}, [\![\ ]\!]_{\mathcal{B}} \rangle$ as follows.

- $W_{\mathcal{B}} = S_{\mathcal{B}}$
- For any $x \in W_{\mathcal{B}}$, $N_{\mathcal{B}}(x) = \{\alpha_a \mid a \in N(x) \textit{ belongs to } \downarrow \Gamma\}$
- For p atomic, $[\![p]\!]_{\mathcal{B}} = \{x \in W_{\mathcal{B}} \mid x : p \textit{ belongs to } \downarrow \Gamma\}$

We now show that $\mathcal{M}_{\mathcal{B}} = \langle W_{\mathcal{B}}, N_{\mathcal{B}}, [\![\ ]\!]_{\mathcal{B}} \rangle$ satisfies the property of non-emptiness for $\mathbb{PCL}$ neighbourhood models: we have to verify that every $\alpha_a \in N(x)$ contains at least one element. If $a \in N(x)$ occurs in the sequent, it must have been introduced either by $\mathsf{R} >$ or $\mathsf{L}|$. If the neighbourhood label has been introduced by $\mathsf{R} >$ or $\mathsf{L}|$, by the saturation conditions associated to both rules it holds that $a \Vdash^{\exists} C$ occurs in $\downarrow \Gamma$. Thus, by the saturation condition $(\mathsf{R} \Vdash^{\exists})$, formula $y \in a$ occurs in Γ.
Moreover, the model $\mathcal{M}_{\mathcal{B}}$ satisfies the following property:

($*$) If $a \subseteq b$ belongs to Γ, then $\alpha_a \subseteq \alpha_b$

To verify ($*$), suppose $y \in \alpha_a$. This means that $y \in a$ belongs to Γ; then, by the saturation condition $L \subseteq$ also $y \in b$ belongs to Γ. By definition of the model we have $y \in \alpha_b$, and thus that $\alpha_a \subseteq \alpha_b$.

Moreover, the model $\mathcal{M}_{\mathcal{B}}$ satisfies the properties of normality, total reflexivity and weak centering.

Normality: To construct a countermodel for logics featuring *only* normality we need to add a clause to the definition of α_a, for $Q = \forall, \exists$:

- If $a \in N(x)$ occurs in Γ, and there are some formulas $a \Vdash^{Q} A$ in $\downarrow \Gamma \cup \downarrow \Delta$,

$$\alpha_a = \{y \in S_{\mathcal{B}} \mid y \in a \textit{ belongs to } \Gamma\};$$

- If $a \in N(x)$ occurs in Γ, but no formulas $a \Vdash^{Q} A$ occur in $\downarrow \Gamma \cup \downarrow \Delta$,

$$\alpha_a = \{w \mid w \textit{ not occurrs in } \Gamma\}.$$

The model satisfies the condition of normality: according to the saturation condition (N), for every x occurring in $\downarrow \Gamma$, there is a such that $a \in N(x)$ occurs in Γ. By definition of $\mathcal{M}_{\mathcal{B}}$, $\alpha_a \in N_{\mathcal{B}}(x)$. Moreover, we have to verify that non-emptiness of the model holds also for the neighbourhood α_a introduced by the rule. If there are some formulas $a \Vdash^{Q} A$ occurring in $\downarrow \Gamma \cup \downarrow \Delta$, the saturation condition associated to either $\mathsf{0}$, $\mathsf{L} \Vdash^{\exists}$ or $\mathsf{R} \Vdash^{\forall}$ ensures that there is at least one formula $y \in a$ in Γ. If there are no such formulas, the application of N is not relevant to the derivation; we introduce an arbitrary world to be placed in the neighbourhood, as specified by the condition added to α_a[12].

Total reflexivity: According to the saturation condition (T), for every x occurring in $\downarrow \Gamma \cup \downarrow \Delta$ also $a \in N(x)$, $x \in a$ occur in Γ. By definition of $\mathcal{M}_{\mathcal{B}}$, this means that $\alpha_a \in N(x)$ and $x \in \alpha_a$, and total reflexivity holds.

Weak centering: Suppose $\alpha_a \in N(x)$. We want to show that $x \in \alpha_a$. By definition, if $\alpha_a \in N(x)$ then $a \in N(x)$ occurs in Γ. By the saturation condition associated to W, it holds that also $x \in a$ occurs in Γ; thus, by definition of the model $x \in \alpha_a$.

Next, define a realization (ρ, σ) such that $\rho(x) = x$ and $\sigma(a) = \alpha_a$ and prove the following claims:

[Claim 1] If $\mathcal{F}$ is in $\downarrow \Gamma$, then $\mathcal{M}_{\mathcal{B}} \models \mathcal{F}$;
[Claim 2] If $\mathcal{F}$ is in $\downarrow \Delta$, then $\mathcal{M}_{\mathcal{B}} \not\models \mathcal{F}$;

where $\mathcal{F}$ denotes a labelled formula, i.e., $\mathcal{F}$ is $a \in N(x)$, $x \in a$, $a \subseteq b$, $x \Vdash^{\forall} A$, $x \Vdash^{\exists} A$, $x \Vdash_a A|B$, $x : A$, $x : A > B$. The two claims are proved by cases, by induction on the weight of the formula $\mathcal{F}$.

[a] If $\mathcal{F}$ is a formula of the form $a \in N(x)$, $x \in a$ or $a \subseteq b$, Claim 1 holds by definition of $\mathcal{M}_{\mathcal{B}}$, and Claim 2 is empty. For the case of $a \subseteq b$, employ the fact $(*)$ above.

[b] If A is a labelled atomic formula $x : p$, the claims hold by definition of the model; by the saturation condition associated to init no inconsistencies arise. If $A \equiv \bot$, the formula is not forced in any model and Claim 2 holds, while Claim 1 holds by the saturation clause associated to $\bot_{\mathsf{L}}$. If A is a conjunction, disjunction or implication, both claims hold for the corresponding saturation conditions and by inductive hypothesis on formulas on smaller weight.

[c] If $A \equiv a \Vdash^{\exists} A$ is in $\downarrow \Gamma$, then by the saturation clause associated to $\mathsf{L} \Vdash^{\exists}$ for some x there are $x \in a$, $x : A$ are in $\downarrow \Gamma$. By definition of the model $\mathcal{M}_{\mathcal{B}}$, for some x, $x \in \alpha_a$. Then, since $w(x : A) < w(a \Vdash^{\exists} A)$, apply the inductive hypothesis and obtain $\mathcal{M}_{\mathcal{B}} \models x : A$. Therefore, by definition of satisfiability, $\mathcal{M}_{\mathcal{B}} \models \alpha_a \Vdash^{\exists} A$.
If $a \Vdash^{\exists} A$ is in $\downarrow \Delta$, then it is also in Δ. Consider an arbitrary world x in α_a. By definition of $\mathcal{M}_{\mathcal{B}}$ we have that $x \in a$ is in Γ; apply the saturation condition associated to $\mathsf{R} \Vdash^{\forall}$ and obtain that $x : A$ is in $\downarrow \Delta$. By inductive hypothesis, $\mathcal{M}_{\mathcal{B}} \not\models x : A$; thus, since this line of reasoning holds for arbitrary x, we can conclude by definition of satisfiability that $\mathcal{M}_{\mathcal{B}} \not\models \alpha_a \Vdash^{\exists} A$. The case in which $A \equiv a \Vdash^{\forall} A$ is similar.

[d] If $x \Vdash_a A|B$ is in $\downarrow \Gamma$, then by the saturation condition associated to $\mathsf{L}|$ for some c it holds that $c \in N(x)$ and $c \subseteq a$ are in Γ, and $a \Vdash^{\exists} A$, $a \Vdash^{\forall} A \to B$ are in $\downarrow \Gamma$.

[12] There is no need to verify non-emptiness for stronger conditions of total reflexivity and weak centering, since the rules added to the calculus add a world belonging to the neighbourhood introduced.

By definition of the model, $\alpha_c \subseteq \alpha_a$, and by inductive hypothesis $\mathcal{M}_{\mathcal{B}} \vDash \alpha_c \Vdash^{\exists} A$ and $\mathcal{M}_{\mathcal{B}} \vDash \alpha_c \Vdash^{\forall} A \to B$. By definition, this yields $\mathcal{M}_{\mathcal{B}} \vDash x \Vdash_a A|B$.
If $x \Vdash_a A|B$ is in $\downarrow \Delta$, consider a neighbourhood $\alpha_c \subseteq \alpha_a$ in $N(x)$. Then by definition of $\mathcal{M}_{\mathcal{B}}$ we have that $c \in N(x)$ and $c \subseteq a$ are in Γ; apply the saturation condition associated to $\mathsf{R}|$ and obtain that either $c \Vdash^{\exists} A$ or $c \Vdash^{\forall} A \to B$ is in $\downarrow \Delta$. By inductive hypothesis, either $\mathcal{M} \nvDash \alpha_c \Vdash^{\exists} A$ or $\mathcal{M}_{\mathcal{B}} \nvDash \alpha_c \Vdash^{\forall} A \to B$. In both cases, by definition $\mathcal{M}_{\mathcal{B}} \nvDash x \Vdash_a A|B$.

[e] If $x : A > B$ is in $\downarrow \Gamma$, then it is also in Γ. Consider an arbitrary neighbourhood α_a in $N(x)$. By definition of $\mathcal{M}_{\mathcal{B}}$ we have that $a \in N(x)$ is in Γ; apply the saturation condition associated to $\mathsf{L} >'$ and conclude that either $a \Vdash^{\exists} A$ is in $\downarrow \Delta$, or $x \Vdash_a A|B$ is in $\downarrow \Gamma$. By inductive hypothesis, it holds that either $\mathcal{M}_{\mathcal{B}} \nvDash \alpha_a \Vdash^{\exists} A$ or $\mathcal{M}_{\mathcal{B}} \vDash x \Vdash_a A|B$. In both cases, by definition $\mathcal{M}_{\mathcal{B}} \vDash x : A > B$.
If $x : A > B$ is in $\downarrow \Delta$, by the saturation condition associated to $\mathsf{R} >$, for some a it holds that $a \in N(x)$ is in Γ, $a \Vdash^{\exists} A$ is in $\downarrow \Gamma$ and $x \Vdash_a A|B$ is in $\downarrow \Delta$. By inductive hypothesis, $\mathcal{M}_{\mathcal{B}} \vDash \alpha_a \Vdash^{\exists} A$ and $\mathcal{M}_{\mathcal{B}} \nvDash x \Vdash_a A|B$, thus, by definition, we have $\mathcal{M}_{\mathcal{B}} \nvDash x : A > B$.

To construct a countermodel for systems with *strong centering* we need to change the construction of the model in order to account for new formulas in which neighbourhood $\{x\}$ occurs. The idea is that, in this case, worlds of the countermodel are not defined as the set $S_{\mathcal{B}}$ of labels occurring in the branch, but as *equivalence classes* $[x]$ with respect to the relation $y \in \{x\}$, which we will show to be an equivalence relation. Then, we require $[x]$ to be contained in any neighbourhood of $N(x)$. For $S_{\mathcal{B}}$, $N_{\mathcal{B}}$ and α_a as defined before, let

$[x] = \{y \in S_{\mathcal{B}} \mid y \in \{x\} \textit{ occurs in } \Gamma\}$;
$[x] \subseteq \alpha_a$, for $a \in N(x)$ occurring in Γ.

We construct a model $\mathcal{M}^c_{\mathcal{B}} = \langle W^c, N^c, [\![\ ]\!]^c \rangle$ as follows:

- $W^c = \{[x] \mid x \in S_{\mathcal{B}}\}$;
- for each $[x] \in W^c$, $N^c([x]) = \{\alpha_a \mid a \in N(x) \textit{ belongs to } \downarrow \Gamma\}$;
- for p atomic, $[\![p]\!]^c = \{[x] \in W^c \mid x : p \textit{ belongs to } \downarrow \Gamma\}$.

We first prove that $y \in \{x\}$ is an equivalence relation. The relation is *reflexive*: for each x occurring in Γ, $x \in \{x\}$ occurs in Γ. This holds from the saturation conditions associated to N, C and Single. To prove that the relation is *symmetric*, we have to show that if $y \in \{x\}$ occurs in Γ, then also $x \in \{y\}$ occurs in Γ. By reflexivity, we have that $y \in \{y\}$. Thus, by the saturation condition associated to Repl_2, we have that also $x \in \{y\}$ belongs to Γ. To prove the converse, use saturation condition associated to Repl_1. To show that the relation is *transitive* we have to prove that if $y \in \{x\}$ and $x \in \{z\}$ occur in Γ, also $y \in \{z\}$ occurs in Γ. By saturation conditions N and C and Single, we have that also $\{z\} \in N(z)$ occurs in Γ. By the saturation condition associated to Repl_1 applied to $x \in \{z\}$, also $\{z\} \in N(x)$ occurs in the sequent; thus, by the saturation condition associated to C we have that both formulas $\{x\} \subseteq \{z\}$ and $\{x\} \in N(z)$ occur in Γ. Finally, by the saturation condition associated to $\mathsf{L} \subseteq$, since $y \in \{x\}$ and $\{x\} \subseteq \{z\}$, we have that $y \in \{z\}$ occurs in the sequent.

Next we need to show that the definitions of $N^c([x])$ and $[\![p]\!]^c$ do not depend on the chosen representative of the equivalence class in question.

i) if $y \in [x]$, then $a \in N(x)$ is in Γ if and only if $a \in N(y)$ is in Γ;
ii) if $y \in [x]$, then $x : p$ is in Γ if and only if $y : p$ is in Γ.

Fact *i)* follows from the saturation conditions associated to Repl_1 and Repl_2, applied to on the formulas $a \in N(x)$ and $a \in N(y)$. Fact *ii)* follows from application of the same saturation conditions to $x : p$ and $y : p$.

The model $\mathcal{M}^c_{\mathcal{B}}$ satisfies the property of centering. Observe that in our model $\{x\}$ corresponds to $[x]$: both are defined as the set containing exactly one element, x. Suppose $\alpha_a \in N(x)$; we have to show that $\{x\} \subseteq \alpha_a$ and $\{x\} \in N(x)$. By definition of the model we have that $[x] \subseteq \alpha_a$, and from this and $\alpha_a \in N(x)$ it follows that $[x] \in N([x])$; thus, strong centering holds.
The following facts are needed in the proof Claims 1 and 2 below.

1) If $a \subseteq b$ belongs to Γ, then $\alpha_a \subseteq \alpha_b$;
2) if $[x] \in [\![A]\!]$ and $y \in [x]$, then $[y] \in [\![A]\!]$;
3) If $[x] \in [\![A]\!]$, then $x : A$ belongs to $\downarrow \Gamma$.

Fact 1) is proved in the same way as $(*)$; proof of 2) and 3) are immediate from admissibility of Repl_1 and Repl_2 in their generalized form.

Finally, we define define a realization (ρ, σ) such that $\rho(x) = [x]$ and $\sigma(a) = \alpha_a$, and prove that:

[Claim 1] If $\mathcal{F}$ is in $\downarrow \Gamma$, then $\mathcal{M}^c_{\mathcal{B}} \vDash \mathcal{F}$;
[Claim 2] If $\mathcal{F}$ is in $\downarrow \Delta$, then $\mathcal{M}^c_{\mathcal{B}} \nvDash \mathcal{F}$.

Again, $\mathcal{F}$ denotes the labelled formulas of the language, including $y \in \{x\}$, $\{x\} \in N(x)$, $\{x\} \in a$. The two cases are proved by distinction of cases, and by induction on the height of the derivation. If $\mathcal{F}$ is a relational formula that does not contain any singleton, Claim 1 holds by definition of the model, and Claim 2 is empty as in case a) of proof of the previous models. Similarly, if $\mathcal{F}$ is either $y \in \{x\}$, $\{x\} \in N(x)$ or $\{x\} \subseteq a$, Claim 1 is satisfied by definition.

The cases b)- e) of the previous proof remain unchanged; condition 2) ensures that all the elements of an equivalence class of world labels satisfy the same sets of formulas. □

Completeness of the calculus is an obvious consequence:

Theorem 3.6.2 (Completeness)**.** If A is valid in $\mathbb{PCL}$, $\mathbb{PCLN}$, $\mathbb{PCLT}$, $\mathbb{PCLW}$ or $\mathbb{PCLC}$, then sequent $\Rightarrow x : A$ is derivable in **G3CL**, $\mathbf{G3CL^N}$, $\mathbf{G3CL^T}$, $\mathbf{G3CL^W}$ or $\mathbf{G3CL^C}$, respectively.

Theorem 3.6.1 together with the soundness of $\mathbf{G3CL}^*$ provides a constructive proof of the finite model property for the logics: if A is valid in a model (meaning that $\neg A$ is not valid), by soundness of the calculi $\neg A$ is not provable. Thus by Theorem 3.6.1 we build a finite countermodel of $\neg A$, i.e. a finite model for A.

3.7 Nesting

In the previous sections we defined the rules of a sound and complete family of labelled sequent calculi which capture in a modular way all conditional logics extending $\mathbb{PCL}$. However, leaving aside the requirement of modularity, it is possible to define significantly simpler versions of calculi $\mathbf{G3CL}^*$ for logics with specific axioms. In this section we introduce a family of alternative sequent calculi for logics with nesting, i.e., for the logics introduced by Lewis in [57]. In Section 3.8 is presented a family of sequent calculi adapted for logics with absoluteness.

The nesting condition states that, given two neighbourhoods α and β in the same system of neighbourhood $N(x)$, either α is included in β or β is included in α. According to Lewis, this condition is essential to formalize counterfactuals: each sphere represents a degree of similarity to the actual world, and sphere nesting ensures that two worlds in the same system of spheres are always comparable with respect to similarity. Nested neighbourhood are "spheres", the object Lewis defined in [57].

The condition of nesting allows for a substantial simplification of the sequent calculus $\mathbf{G3CL^V}$ and its extensions. First, we reformulate the language by introducing the operator of comparative plausibility, $\preccurlyeq$, as primitive.

Definition 3.7.1. The set of well formed formulas of $\mathbb{V}$ and its extension is defined by means of the following grammar, for $p \in \mathcal{V}$ propositional variable and $A, B \in \mathcal{F}^{\preccurlyeq}$:

$$\mathcal{F}^{\preccurlyeq} ::= p \mid \bot \mid A \to B \mid A \preccurlyeq B.$$

The operator $\preccurlyeq$ denotes comparative plausibility, with $A \preccurlyeq B$ read as "A is at least as plausible as B". The two operators $>$ and $\preccurlyeq$ are interdefinable by means of the following equivalences (valid in all logics):

$$\begin{array}{lrcl} (1) & A > B & \equiv & (\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \wedge B)) \\ (2) & A \preccurlyeq B & \equiv & ((A \vee B) > \bot) \vee \neg((A \vee B) > \neg A) \end{array}$$

The truth condition of the comparative plausibility operator in nested neighbourhood models is the following:

$$x \Vdash A \preccurlyeq B \quad \textit{iff} \quad \textit{for all } \alpha \in N(x), \textit{ if } \alpha \Vdash^{\exists} B \textit{ then } \alpha \Vdash^{\exists} A.$$

Lewis introduced the connective of comparative plausibility in [57], and gave an axiomatization of his logics in terms of this operator as follows.

$$\begin{array}{c|ll} \mathbb{V} & & \text{Axiomatization of classical propositional logic} \\ & \text{(CPR)} & \dfrac{\vdash B \to A}{\vdash A \preccurlyeq B} \\ & \text{(CPA)} & (A \preccurlyeq (A \vee B)) \vee (B \preccurlyeq (A \vee B)) \\ & \text{(TR)} & (A \preccurlyeq B) \wedge (B \preccurlyeq C) \to (A \preccurlyeq C) \\ & \text{(CO)} & (A \preccurlyeq B) \vee (B \preccurlyeq A) \end{array}$$

$$\begin{array}{c|ll} \text{Extensions} & & \text{Axiomatization of } \mathbb{V} \\ \text{of } \mathbb{V} & \text{(N)} & \neg(\bot \preccurlyeq \top) \\ & \text{(T)} & (\bot \preccurlyeq \neg A) \to A \\ & \text{(W)} & A \to (A \preccurlyeq \top) \\ & \text{(C)} & (A \preccurlyeq \top) \to A \end{array}$$

Figure 3.4: Lewis' logics and axioms.

Remark 3.7.1. The truth condition for $\preccurlyeq$ in models without the condition of nesting on neighbourhoods is the following:

> $(*)$ For all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} B$ then there exists $\beta \in N(x)$ such that $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} A$.

Since $(*)$ is composed of an existential quantifier under the scope of a universal quantifier, this condition is as rich as the truth condition for the conditional operator. In nested models the truth condition for $\preccurlyeq$ becomes significantly simpler; it is thus convenient to define an equivalent version of the labelled sequent calculus by means of the comparative plausibility operator.

We here define labelled sequent calculi for $\mathbb{V}$, $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$ and $\mathbb{VC}$. We call $\mathbf{G3V}$ (instead of $\mathbf{G3CL^V}$) the sequent calculus for $\mathbb{V}$ and, as before, we define its extensions as follows: $\mathbf{G3V^N} = \mathbf{G3V} + \mathsf{N} + \mathsf{0}$; $\mathbf{G3V^T} = \mathbf{G3V^N} + \mathsf{T}$; $\mathbf{G3V^W} = \mathbf{G3V^T} + \mathsf{W}$ and

Initial sequents and propositional rules: as **G3CL**

Rules for local forcing

$$\frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ \mathsf{L} \Vdash^{\exists}\ (\mathsf{x!}) \qquad \frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{R} \Vdash^{\exists}$$

Rules for comparative plausibility

$$\frac{a \Vdash^{\exists} B, a \in N(x), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : A \preccurlyeq B}\ \mathsf{R} \preccurlyeq (\mathsf{a!})$$

$$\frac{a \in N(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} B \quad a \Vdash^{\exists} A, a \in N(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta}{a \in N(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta}\ \mathsf{L} \preccurlyeq$$

Rules for inclusion and nesting

$$\frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ \mathsf{L} \subseteq$$

$$\frac{a \subseteq b, a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta \qquad b \subseteq a, a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{Nes}$$

Rules for extensions

$$\frac{a \in N(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{N}\ (\mathsf{a!}) \qquad \frac{y \in a, a \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{0}\ (\mathsf{y!})(\star)$$

$$\frac{x \in a, a \in N(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{T}\ (\mathsf{a!}) \qquad \frac{x \in a, a \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{W}$$

$$\frac{x \in \{x\}, \{x\} \in N(x), \Gamma \Rightarrow \Delta}{\{x\} \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{Single} \qquad \frac{\{x\} \in N(x), \{x\} \subseteq a, a \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{C}$$

$$\frac{y \in \{x\}, At(x), At(y), \Gamma \Rightarrow \Delta}{y \in \{x\}, At(x), \Gamma \Rightarrow \Delta}\ \mathsf{Repl}_1\ (*) \qquad \frac{y \in \{x\}, At(x), At(y), \Gamma \Rightarrow \Delta}{y \in \{x\}, At(y), \Gamma \Rightarrow \Delta}\ \mathsf{Repl}_2\ (*)$$

$(\star)$ $w \in a$ does not occur in Γ; $a \Vdash^{\exists} A$ or $a \Vdash^{\forall} A$ occur in $\Gamma \cup \Delta$.

Figure 3.5: Rules of **G3V**

$\mathbf{G3V^C} = \mathbf{G3V^W} + \mathsf{C} + \mathsf{Single} + \mathsf{Repl}_1 + \mathsf{Repl}_2$. We write $\mathbf{G3V}^*$ to denote the family of nested sequent calculi.The rules of $\mathbf{G3V}^*$ are given in Figure 3.5. Observe that it is not necessary to define rules $\mathsf{R}|$ and $\mathsf{L}|$, since the two rules $\mathsf{L} \preccurlyeq$ and $\mathsf{R} \preccurlyeq$ suffice to express the truth condition of $\preccurlyeq$. Furthermore, there is no need to include in the calculus rules $\mathsf{R} \Vdash^{\forall}$ and $\mathsf{L} \Vdash^{\forall}$, since no formula $a \Vdash^{\forall} A$ is introduced (bottom-up) in the derivation. The rules of Ref and Tr will be proved admissible in Lemma 3.7.4 and Lemma 3.7.5.
To prove soundness of the sequent calculi, we need to define the notion of realization. This is done as in Definition 3.3.3, adding the following clause for $\preccurlyeq$:

> $\mathcal{M} \vDash_{\rho,\sigma} x \Vdash A \preccurlyeq B$ if for all $\sigma(a) \in N(\rho(x))$, if $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} B$ then $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$.

Theorem 3.7.1 (Soundness)**.** If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathbf{G3V}^*$, then it is valid in all sphere models.

Proof. Straightforward, by induction on the height of the derivation. □

Derivability of generalized initial sequents and of generalized versions of the rules of replacement can be proved as for sequent calculus **G3CL**. Moreover, the calculi $\mathbf{G3V}^*$ enjoy admissibility of weakening and contraction, and the proof is similar to the proofs for **G3CL***. As for cut admissibility, the theorem is proved as in Section 3.4, adding the cases for rules $\mathsf{R}\preccurlyeq$ and $\mathsf{L}\preccurlyeq$.

Theorem 3.7.2. The rule of cut is admissible in **G3V***.

Proof. By primary induction on the weight of formulas and secondary induction on the sum of heights of the derivations of the premisses. The proof is by cases, according to the last rules applied in the derivations of the premisses of cut. We show only the case in which both occurrences of the cut formula are principal and derived by $\mathsf{R}\preccurlyeq$ and $\mathsf{L}\preccurlyeq$ respectively. We have:

$$\dfrac{\dfrac{\overset{(1)}{b \in N(x), b \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, b \Vdash^{\exists} A}}{\Gamma \Rightarrow \Delta, x : A \preccurlyeq B}\ \mathsf{R}\preccurlyeq \quad \dfrac{(2) \qquad (3)}{a \in N(x), x : A \preccurlyeq B, \Gamma' \Rightarrow \Delta'}\ \mathsf{L}\preccurlyeq}{\Gamma, a \in N(x), \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

where $(2) = a \in N(x), x : A \preccurlyeq B, \Gamma' \Rightarrow \Delta', a \Vdash^{\exists} B$ and $(3) = a \Vdash^{\exists} A, a \in N(x), x : A \preccurlyeq B, \Gamma' \Rightarrow \Delta'$ are the left and right premisses of $\mathsf{L}\preccurlyeq$. In (1), substitute variable b with variable a. The derivation is transformed as follows:

$$\dfrac{\dfrac{\overset{(2)}{a \in N(x), x : A \preccurlyeq B, \Gamma' \Rightarrow \Delta', a \Vdash^{\exists} B} \quad \dfrac{(1)[a/b] \qquad (3)}{\Gamma, a \Vdash^{\exists} B, a \in N(x), \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}}{\Gamma, a \in N(x), a \in N(x), \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}\ \mathsf{cut}}{\Gamma, a \in N(x), \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{Ctr}$$

where the upper occurrence of cut has smaller sum of heights than the original cut, and the lower occurrence of cut is applied to formulas of smaller weight than the cut formula. □

Theorem 3.7.3. If a formula A is valid in $\mathbb{V}$ and its extensions, the sequent $\Rightarrow x : A$ is derivable in **G3V***.

Proof. Refer to Appendix 3.A at the end of this chapter. □

Lemma 3.7.4. The rule of reflexivity is height-preserving admissible in **G3V***: If $a \subseteq a, \Gamma \Rightarrow \Delta$ is derivable, then $\Gamma \Rightarrow \Delta$ is derivable with no greater derivation height.

Proof. By induction on the height h of the derivation. If $h = 0$, $a \subseteq a, \Gamma \Rightarrow \Delta$ is an instance of init or $\bot_{\mathsf{L}}$; then, so is $\Gamma \Rightarrow \Delta$. If $h > 0$, we look at the last rule applied. All cases are straightforward by application of the inductive hypothesis to the premiss(es). Consider for instance rule $\mathsf{L}\subseteq$, in case $b = a$:

$$\dfrac{y \in a, a \subseteq a, y \in a, \Gamma' \Rightarrow \Delta}{a \subseteq a, y \in a, \Gamma' \Rightarrow \Delta}\ \mathsf{L}\subseteq$$

We apply the inductive hypothesis to the premiss; then, contraction on $y \in a$ yields the desired endsequent.

□

Lemma 3.7.5. The rule of transitivity is admissible in **G3V***: If a sequent $a \subseteq c, a \subseteq b, b \subseteq c, \Gamma \Rightarrow \Delta$ is derivable, then $a \subseteq b, b \subseteq c, \Gamma \Rightarrow \Delta$ is derivable.

Proof. Induction on the height h of the derivation of sequent $a \subseteq c, a \subseteq b, b \subseteq c, \Gamma \Rightarrow \Delta$. Assume that the sequent is derivable. If $h = 0$, the sequent is an instance of init or of $\bot_{\mathsf{L}}$; then, so is sequent $a \subseteq b, b \subseteq c, \Gamma \Rightarrow \Delta$. If $h > 0$, we distinguish cases according to the last rule applied to derive the sequent. If the last rule applied is a rule that does not involve inclusions (thus a propositional rule, a $\preccurlyeq$-rule, $\mathsf{R}\Vdash^{\exists}$ and $\mathsf{L}\Vdash^{\exists}$), the proof is straightforward by applying inductive hypothesis to the premiss(es) of the rule. Thus, the only relevant case are those in which the last rule applied is either $\mathsf{L} \subseteq$ or Nes. Let us consider $\mathsf{L} \subseteq$. The derivation of the premiss of Tr is the following:

$$\dfrac{y \in c, a \subseteq c, a \subseteq b, b \subseteq a, y \in a, \Gamma' \Rightarrow \Delta}{a \subseteq c, a \subseteq b, b \subseteq a, y \in a, \Gamma' \Rightarrow \Delta}\ \mathsf{L} \subseteq$$

By inductive hypothesis, the sequent $y \in c, a \subseteq b, b \subseteq a, y \in a, \Gamma' \Rightarrow \Delta$ is derivable. We apply the following transformation:

$$\dfrac{\dfrac{\dfrac{y \in c, a \subseteq b, b \subseteq c, y \in a, \Gamma' \Rightarrow \Delta}{y \in b, y \in c, a \subseteq b, b \subseteq c, y \in a, \Gamma' \Rightarrow \Delta}\ \mathsf{Wk}}{y \in b, a \subseteq b, b \subseteq c, y \in a, \Gamma' \Rightarrow \Delta}\ \mathsf{L} \subseteq}{a \subseteq b, b \subseteq c, y \in a, \Gamma' \Rightarrow \Delta}\ \mathsf{L} \subseteq$$

Now suppose the premiss of Tr has been derived by Nes.

$$\dfrac{a \subseteq c, a \subseteq c, a \subseteq b, b \subseteq c, \Gamma' \Rightarrow \Delta \quad c \subseteq a, a \subseteq c, a \subseteq b, b \subseteq c, \Gamma' \Rightarrow \Delta}{a \subseteq c, a \subseteq b, b \subseteq c, \Gamma' \Rightarrow \Delta}\ \mathsf{Nes}$$

It suffices to apply the inductive hypothesis to the two premisses, and apply Nes again. □

To prove termination for **G3V** and its extensions we employ the same strategy followed for **G3CL**. Thus, we define the notion of saturated sequents (adding the clauses for the new rules) and describe a terminating proof search strategy.

Definition 3.7.2. Let $\mathcal{B} = S_0, S_1, \ldots$ be a derivation branch with S_i sequents $\Gamma_i \Rightarrow \Delta_i$ for $i = 1, 2, \ldots$ and S_0 the sequent $\Rightarrow x : A_0$. A sequent $\Gamma \Rightarrow \Delta$ *satisfies the saturation condition with respect to a rule* R if the corresponding saturation condition for the rule holds. The saturation conditions for most of the rules can be found in Definition 3.5.2. As for the new rules,

($\mathsf{L} \preccurlyeq$) If $x : A \preccurlyeq B$ and $a \in N(x)$ occur in Γ, then either $a \Vdash^{\exists} B$ occurs in $\downarrow \Delta$ or $a \Vdash^{\exists} A$ occurs in Γ;

($\mathsf{R} \preccurlyeq$) If $x : A \preccurlyeq B$ occurs in $\downarrow \Delta$, then $a \Vdash^{\exists} B$ occurs in $\downarrow \Gamma$ and $a \Vdash^{\exists} A$ occurs in Δ.

Definition 3.7.3. The proof search strategy consists of clauses (ii) - (v) from Definition 3.5.3.

To check the validity of a formula A, we try to build a derivation with root $\Rightarrow x : A$ according to the strategy. We show that the process of building such a derivation always comes to an end in a finite number of steps.

We define an acyclic graph as in Definition 3.5.4 with the world and neighbourhood labels that could occur in any derivation starting with $\Rightarrow x : A_0$. As in Lemma 3.5.3, it holds that the relation $\xrightarrow{w}$ does not contain any cycles and forms a graph displaying at the root the world label x_0. This can be verified by by inspection on the rules. Moreover, all labels occurring in a derivation branch also occur in the associated graph (cf. proof as in Lemma 3.5.3). We have to prove that the acyclic graph contains a finite number of elements.

Lemma 3.7.6. Given a saturated branch $\mathcal{B}$ and a world label x, the size of $\mathsf{Neigh}(x)$ and $\mathsf{World}(x)$ is finite: $|\mathsf{Neigh}(x)| = O(n)$ and $|\mathsf{World}(x)| = O(n^2)$.

Proof. We first prove that $|\mathsf{Neigh}(x)| = O(n)$, i.e., that each world label generates a finite number of neighbourhood labels. By definition, $x \to_g a$ if there exists a t such that a does not occur in Γ_s for all $s < t$ and $a \in N(x)$ occurs in Γ_t. This means that label a has been introduced in the derivation either by some rule for semantic conditions or by $\mathsf{R} \preccurlyeq$.

In case of rules for semantic conditions, at most $O(n)$ neighbourhood labels are introduced (cf. proof of Lemma 3.5.4). In case of $\mathsf{R} \preccurlyeq$, formula $a \in N(x)$ is generated from application of the rule to some formula $x : C \preccurlyeq D$ occurring in $\downarrow \Delta_s$. The number of such formulas is $O(n)$, where $n = d(A_0)$. Moreover, since $x : C \preccurlyeq D$ disappears from the conclusion to the premiss(es), we have that at most $O(n)$ neighbourhood labels are introduced.

We now prove $|\mathsf{World}(x)| = O(n^2)$. By definition $y \in W(x)$ iff $x \xrightarrow{w} y$, i.e. iff for some b it holds that $x \to_g b$ and $b \to_g y$. As just shown, the number of neighbourhood labels generated by some x is $O(n)$. Let us consider $b \to_g y$. By definition, this means that there exists $t < k$ such that y does not occur in Γ_s for $s \leqslant t$ and $y \in b$ occurs in Γ_{t+1}. A formula $y \in b$ can be introduced in the derivation by either:

- $\mathsf{L} \Vdash^{\exists}$ applied to some $b \Vdash^{\exists} C$ occurring in Γ_t;
- $\mathsf{0}$ applied to some $b \in N(x)$, for some $b \Vdash^{\forall} C$ occurring in Γ_t or $b \Vdash^{\exists} C$ occurring in Δ_t.

In all cases, however, the rule can be applied only once to each formula $b \Vdash^{\exists} C$, $b \Vdash^{\forall} C$ or $b \in N(x)$; moreover, $\mathsf{0}$ is not applied if any of the other two rules can be applied. Thus, the number of neighbourhood labels that can be introduced in the derivation depends on the number of formulas $b \Vdash^{\exists} C$, $b \Vdash^{\forall} C$ that can be introduced, and the number of such formulas is $O(n)$.
formula $b \Vdash^{\exists} C$ could be introduced by either:

a) rule $\mathsf{R} \preccurlyeq$ applied to a formula $x : A \preccurlyeq B$ occurring in $\downarrow \Delta_k$, for $C \equiv B$. In this case, the rule can be applied only *once* to each formula $x : A \preccurlyeq B$ that occurs in the succedent, generating exactly *one* new neighbourhood label. The number of such formulas is $O(n)$.
b) rule $\mathsf{L} \preccurlyeq$ applied to formula $x : A \preccurlyeq B$ occurring in Γ, for a given $b \in N(x)$ and for $C \equiv A$. Again, by the saturation condition, given a sphere label the rule can be applied only once to a formula $x : A > B$ occurring in $\downarrow \Gamma$, and the number of such formulas is $O(n)$.

Thus, each $b \in \mathsf{Neigh}(x)$ generates at most $O(n + 1) = O(n)$ neighbourhood labels. We have shown that $|\mathsf{Neigh}(x)| = O(n)$. The number of world labels that can be generated from a world label x is given by $|\mathsf{World}(x)| = O(n^2)$. $\square$

Lemma 3.7.7. Every branch in the graph generated by the relation $\xrightarrow{w}$ is finite; more precisely, the length of any chain is $O(n)$, where $n = d(A_0)$.

Proof. For any world label x occurring in the branch, and for $d(x)$ conditional degree as in Definition 3.5.5, let

$$d(x) = max\{d(C) \mid x : C \in \downarrow \Gamma \cup \downarrow \Delta\}.$$

By induction on $d(x)$ we show that the length of an arbitrary chain from label x is bounded by the degree of the formula it labels. In symbols, the length of a chain beginning with x is $\leqslant d(x)$.

If $d(x) = 0$, all chains starting from x have length 0, as in proof of Lemma 3.5.5.

If $d(x) > 0$, there must be some $x : C \preccurlyeq D$ occurring in $\downarrow \Gamma \cup \downarrow \Delta$, and there will be at least one chain of length greater than 0 in the graph. Take one of these chains: it will contain a label y as immediate successor of x, and such that $a \in N(x)$ and $y \in a$ both occur in $\downarrow \Gamma \cup \downarrow \Delta$ (this means that label y has been introduced by some combinations of $\mathsf{L} \preccurlyeq$ or $\mathsf{R} \preccurlyeq$ and $\mathsf{L} \Vdash^{\exists}$ or $\mathsf{0}$).

Observe that, in this case, y can occur only as label of formulas which have a smaller degree than $C \preccurlyeq D$. More precisely, for any $C \preccurlyeq D$ such that $x : C \preccurlyeq D$ occurs in $\downarrow \Gamma \cup \downarrow \Delta$, with $d(C \preccurlyeq D) \leqslant d(x)$, and for all formulas $a \Vdash^{\exists} G$ occurring in $\downarrow \Gamma \cup \downarrow \Delta$, with $G = C$ or $G = D$, it holds that $d(G) < d(x)$. Thus, all formulas $y : E$ occurring in $\downarrow \Gamma \cup \downarrow \Delta$ may only be subformulas of a formula G such that $a \Vdash^{\exists} G$. The same holds in case the label y has been generated by $\mathsf{0}$: it might label only subformulas of some conditional formulas. It holds that $d(y) < d(x)$, and more precisely $d(x) = d(y) + 1$. By inductive hypothesis all chains beginning with y have length $\leqslant d(y)$. The chain beginning with x has length $\leqslant (d(y) + 1) \leqslant ((d(x) - 1) + 1) = d(x)$. □

Corollary 3.7.8 (Termination)**.** Let $\Gamma \Rightarrow \Delta$ be a sequent of the form $\Rightarrow x : A_0$. Proof search in **G3V*** always terminates.

Proof. The rules of **G3V*** applied to $\Rightarrow x : A_0$ produce only a finite number of subformulas of A_0. We have to check that the rules do not generate an infinite number of labels. Lemma 3.7.6 and Lemma 3.7.7 ensure that any acyclic graph containing the labels introduced in any derivation branch is finite: the graph has a finite number of immediate successors, and each path in the graph has a finite length. Thus, the number of labels occurring in the derivation is finite, and root-first proof search comes to an end in a finite number of steps. □

Termination yields a decision procedure: to check provability of formula A, build a proof search tree $\mathcal{D}$ with root $\Rightarrow x : A$. By the previous theorem, $\mathcal{D}$ is finite: either every branch of $\mathcal{D}$ terminates with an initial sequent, and $\mathcal{D}$ is a derivation of A, or $\mathcal{D}$ contains an open saturated branch. In the former case A is provable; in the latter case it is not, and it is possible to extract a countermodel of A from the open branch.

As done in the non-nested case, we can give an alternative (semantic) completeness proof for **G3V** and the extensions for which we have proved termination. Again, the following theorem combined with soundness of the calculi provides a constructive proof of the *finite model property* of $\mathbb{V}$.

Theorem 3.7.9 (Completeness)**.** If a formula A is not derivable in **G3V**, **G3VN**, **G3VT**, **G3VW**, **G3VC**, there is a *finite* countermodel for A.

Proof. The proof strategy and the definition of the model are the same as in proof of Theorem 3.6.1. We have to ensure that the model satisfies the condition of nesting. This is done employing the following property, also from Theorem 3.6.1.

$(*)$ If $a \subseteq b$ belongs to Γ, then $\alpha_a \subseteq \alpha_b$

Nesting: Suppose $\alpha_a \in N(x)$ and $\alpha_b \in N(x)$. We want to to show that $\alpha_a \subseteq \alpha_b$ or $\alpha_b \subseteq \alpha_a$. By definition of the model, from $\alpha_a \in N(x)$ and $\alpha_b \in N(x)$ it follows that $a \in N(x)$ and $b \in N(x)$ both belong to Γ. From the saturation condition associated to Nes, we have that $a \subseteq b$ or $b \subseteq a$ belong to Γ and we conclude by fact $(*)$ above. The rest of the proof is exactly as in proof of Theorem 3.6.1. Again, a different model has to be constructed for systems with centering, as described in Theorem 3.6.1. □

3.8 Absoluteness

In this section we present a family of labelled proof systems adapted to capture conditional logics with the condition of absoluteness. With respect to the calculi $\mathbf{G3CL}^*$, the calculi presented in this section modify the structure of sequents, in such a way as to eliminate some relational atoms.

The reason for a reformulation of the labelled systems in case of absoluteness is twofold. On one side, the resulting proof systems are simpler than the corresponding systems in $\mathbf{G3CL}^*$: they display a smaller number of formulas, and derivations tend to be significantly shorter (refer to Appendix 3.A). On the other hand, proving termination for calculi $\mathbf{G3CL}^*$ with absoluteness would require some non-trivial modification of the proof strategy employed to prove termination for the other calculi[13]. Thus, we prove termination of labelled systems with absoluteness for a reformulated version of the calculus.

We define sequent calculi for conditional logic $\mathbb{PCLA}$ and its extensions with normality, total reflexivity, weak centering, centering and nesting. We call $\mathbf{G3A_w}$ the labelled calculus for the basic system, and add letters in correspondence with the semantic conditions to denote sequent calculi for the extensions of $\mathbb{PCLA}$, as specified in Figure 3.6. By $\mathbf{G3A_w^*}$ we denote all the family of sequent calculi here introduced.

Definition 3.8.1. The set of labelled formulas of $\mathbf{G3A_w}^*$ is the same set defined for calculi $\mathbf{G3CL}^*$ (Definition 3.3.1) with the exception of $a \in N(x)$, which is no longer a relational atom for this family of calculi. Moreover, labelled formulas $x \Vdash_a A|B$ are replaced by $a : A|B$, whose interpretation is the following:

$$a : A|B \quad \textit{iff} \quad \textit{there exists } c \subseteq a \textit{ such that } c \Vdash^{\exists} A \textit{ and } c \Vdash^{\forall} A \to B.$$

Sequents in $\mathbf{G3A_w}^*$ have the form

$$\Gamma \Rightarrow_w \Delta$$

where the w indicates that all neighbourhoods $a_1, \dots, a_n$ which might be introduced bottom-up in the derivation belong to the same system of neighbourhoods $N(w)$. Thus, for all neighbourhood labels $a_1, \dots, a_n$ occurring in Γ it holds that $a_1 \in N(w), \dots, a_n \in N(w)$. To prove a formula A, we start with root sequent $\Rightarrow_w w : A$.

The intuitive idea behind this definition is the following. According to the absoluteness condition, for any two worlds $x, y \in W$, the systems of neighbourhoods associated to the worlds is the same: $N(x) = N(y)$. This means that the neighbourhoods belonging to $N(x)$ are the same as the neighbourhoods belonging to $N(y)$. Thus, instead of constructing a system of neighbourhood for each world, we can construct just *one* system of neighbourhood, and assume that all the other systems are the same. In defining the sequents of calculi $\mathbf{G3A_w^*}$ we avoid the use of relational atoms $a \in N(x)$, and assume all neighbourhood labels to belong to the same systems of neighbourhood, the one associated to the arbitrarily chosen world w.

Moreover, since all systems of neighbourhoods are the same, it holds that the comparative plausibility formulas, whose truth conditions depend on the neighbourhood structure, are the same between all the worlds of the model. Thus, the left and a right rule for absoluteness allow conditional formulas to "pass" from some world x to some other world w (Figure 3.6).

The rules of $\mathbf{G3A_w}$ are defined in Figure 3.6. Unless otherwise specified by a proviso on the rule, we assume that the labels occurring in the premiss(es) of a rule already

[13]The reason is that, with the rules of absoluteness, the modal degree of the formula does not necessarily decrease when going from a world label x to the world label y, generated by y. Thus, Lemma 3.5.5 has to be proved in a different way.

Initial sequents

$$\frac{}{x : p, \Gamma \Rightarrow_w \Delta, x : p} \text{ init} \qquad \frac{}{x : \bot, \Gamma \Rightarrow_w \Delta} \bot_{\mathsf{L}}$$

Rules for local forcing

$$\frac{x : A, x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow_w \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow_w \Delta} \text{ L} \Vdash^{\forall}_{\mathsf{A}} \qquad \frac{x \in a, \Gamma \Rightarrow_w \Delta, x : A}{\Gamma \Rightarrow_w \Delta, a \Vdash^{\forall} A} \text{ R} \Vdash^{\exists}_{\mathsf{A}} \text{ (x!)}$$

$$\frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow_w \Delta} \text{ L} \Vdash^{\exists}_{\mathsf{A}} \text{ (x!)} \qquad \frac{x \in a, \Gamma \Rightarrow_w \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow_w \Delta, a \Vdash^{\exists} A} \text{ R} \Vdash^{\exists}_{\mathsf{A}}$$

Propositional rules

$$\frac{w : B, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, w : A}{w : A \to B, \Gamma \Rightarrow \Delta} \text{ L} \to_A \qquad \frac{w : A, \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \to B} \text{ R} \to_A$$

Rules for the conditional and absoluteness

$$\frac{a \Vdash^{\exists} A, \Gamma \Rightarrow_w \Delta, a : A|B}{\Gamma \Rightarrow_w \Delta, w : A > B} \text{ R} >_{\mathsf{A}} \text{ (a!)}$$

$$\frac{w : A > B, \Gamma \Rightarrow_w \Delta, a \Vdash^{\exists} A \quad a : A|B, w : A > B, \Gamma \Rightarrow_w \Delta}{w : A > B, \Gamma \Rightarrow_w \Delta} \text{ L} >_{\mathsf{A}}$$

$$\frac{c \subseteq a, \Gamma \Rightarrow_w \Delta, a : A|B, c \Vdash^{\exists} A \quad c \subseteq a, \Gamma \Rightarrow_w \Delta, a : A|B, c \Vdash^{\forall} A \to B}{c \subseteq a, \Gamma \Rightarrow_w \Delta, a : A|B} \text{ R}|_{\mathsf{A}}$$

$$\frac{c \subseteq a, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B, \Gamma \Rightarrow_w \Delta}{a : A|B, \Gamma \Rightarrow_w \Delta} \text{ L}|_{\mathsf{A}} \text{ (c!)}$$

$$\frac{w : A > B, \Gamma \Rightarrow_w \Delta}{x : A > B, \Gamma \Rightarrow_w \Delta} \text{ LA } (\mathsf{x} \neq \mathsf{w}) \qquad \frac{\Gamma \Rightarrow_w \Delta, w : A > B}{\Gamma \Rightarrow_w \Delta, x : A > B} \text{ RA } (\mathsf{x} \neq \mathsf{w})$$

Rules for inclusion

$$\frac{a \subseteq a, \Gamma \Rightarrow_w \Delta}{\Gamma \Rightarrow_w \Delta} \text{ Ref}_{\mathsf{A}} \quad \frac{c \subseteq a, c \subseteq b, b \subseteq a, \Gamma \Rightarrow_w \Delta}{c \subseteq b, b \subseteq a, \Gamma \Rightarrow_w \Delta} \text{ Tr}_{\mathsf{A}} \quad \frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow_w \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow_w \Delta} \text{ L} \subseteq_{\mathsf{A}}$$

Rules for extensions

$$\frac{a \subseteq a, \Gamma \Rightarrow_w \Delta}{\Gamma \Rightarrow_w \Delta} \text{ N}_{\mathsf{A}} \text{ (a!)} \qquad \frac{y \in a, \Gamma \Rightarrow_w \Delta}{\Gamma \Rightarrow_w \Delta} \text{ 0}_{\mathsf{A}} \text{ (y!)}(\star)$$

$$\frac{w \in a, \Gamma \Rightarrow_w \Delta}{\Gamma \Rightarrow_w \Delta} \text{ T}_{\mathsf{A}} \text{ (a!)} \quad \frac{w \in a, \Gamma \Rightarrow_w \Delta}{\Gamma \Rightarrow_w \Delta} \text{ W}_{\mathsf{A}} \quad \frac{w \in \{w\}, \Gamma \Rightarrow_w \Delta}{\{w\} \subseteq a, \Gamma \Rightarrow_w \Delta} \text{ Single}_{\mathsf{A}}$$

$$\frac{\{w\} \subseteq a, \Gamma \Rightarrow_w \Delta}{\Gamma \Rightarrow_w \Delta} \text{ C}_{\mathsf{A}} \qquad \frac{a \subseteq \{w\}, \{w\} \subseteq a, \Gamma \Rightarrow \Delta}{\{w\} \subseteq a, \Gamma \Rightarrow \Delta} {\text{C}_{\mathsf{A}}}^*$$

$$\frac{y \in \{x\}, At(x), At(y), \Gamma \Rightarrow_w \Delta}{y \in \{x\}, At(x), \Gamma \Rightarrow_w \Delta} \text{ Repl}_{\mathsf{A1}} \ (*) \qquad \frac{y \in \{x\}, At(x), At(y), \Gamma \Rightarrow_w \Delta}{y \in \{x\}, At(y), \Gamma \Rightarrow_w \Delta} \text{ Repl}_{\mathsf{A2}} \ (*)$$

$$\frac{a \subseteq b, \Gamma \Rightarrow_w \Delta \quad b \subseteq a, \Gamma \Rightarrow_w \Delta}{\Gamma \Rightarrow_w \Delta} \text{ Nes}_{\mathsf{A}}$$

($\star$) $w \in a$ does not occur in Γ; $a \Vdash^{\exists} A$ or $a \Vdash^{\forall} A$ occur in $\Gamma \cup \Delta$.
($*$) $At(x) := x : p, x \in a, y \in \{x\}$ for p atomic formula.

$\mathbf{G3AN_w} = \mathbf{G3A_w} + \mathsf{N_A} + \mathsf{0_A}$; $\mathbf{G3AT_w} = \mathbf{G3AN_w} + \mathsf{T_A}$; $\mathbf{G3AW_w} = \mathbf{G3AT_w} + \mathsf{W_A}$;
$\mathbf{G3AC_w} = \mathbf{G3AW_w} + \mathsf{C_A} + \mathsf{C_A}^* + \mathsf{Single_A} + \mathsf{Repl_{A1}} + \mathsf{Repl_{A2}}$;
$\mathbf{G3AV_w} = \mathbf{G3A_w} + \mathsf{Nes_A}$; $\mathbf{G3AVN/T/W/C_w} = \mathbf{G3AN/T/W/C} + \mathsf{Nes_A}$

Figure 3.6: Rules of $\mathbf{G3A^*_w}$

occur in the conclusion. Thus, for instance, in rule $\mathsf{Single_A}$ the neighbourhood $\{x\}$ should already occur in the conclusion (most probably in the form $\{x\} \subseteq a$), whereas rule C introduces $\{x\}$ in the derivation, as specified by the side condition. Compare rule $\mathsf{N_A}$ with rule N of Figure 6.2, which introduces a new neighbourhood label without any other formula. We have added formula $a \subseteq a$ in the premiss of $\mathsf{N_A}$, in order to add a neighbourhood label in the new calculus, which does not admit formulas $a \in N(w)$.

Remark 3.8.1. Logics $\mathbb{PCLAC}$ and $\mathbb{VA}$ (its extension with nesting) collapse into classical logic. Thus, in $\mathbf{G3AC_w}$ and $\mathbf{G3AVC_w}$ the following formulas are derivable:

a) $A > B \to (A \to B)$
b) $(A \to B) \to A > B$

Formula a) is derivable by means of the rules $\mathsf{N_A}$ and $\mathsf{W_A}$; thus, is already derivable in both $\mathbf{G3AC_w}$ and $\mathbf{G3AVC_w}$.

$$
\begin{prooftree}
\AxiomC{$w \in a \ldots w : \overset{\triangledown}{A} \Rightarrow_w \ldots, w : A$}
\RightLabel{$\mathsf{R} \Vdash^{\exists}_{\mathsf{A}}$}
\UnaryInfC{$w \in a \ldots w : A \Rightarrow_w w : B, a \Vdash^{\exists} A$}
\RightLabel{$\mathsf{W_A}$}
\UnaryInfC{$a \subseteq a, w : A > B, w : A \Rightarrow_w w : B, a \Vdash^{\exists} A$}
\RightLabel{$\mathsf{R} \to_A$}
\UnaryInfC{$a \subseteq a, w : A > B \Rightarrow_w w : A \to B, a \Vdash^{\exists} A$}
\AxiomC{$w \in c, w : A \to \overset{\triangledown}{B} \cdots \Rightarrow_w w : A \to B$}
\RightLabel{$\mathsf{L} \Vdash^{\forall}$}
\UnaryInfC{$w \in c, c \Vdash^{\forall} A \to B \cdots \Rightarrow_w w : A \to B$}
\RightLabel{$\mathsf{W_A}$}
\UnaryInfC{$c \subseteq a, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B \cdots \Rightarrow_w w : A \to B$}
\RightLabel{$\mathsf{L}|_{\mathsf{A}}$}
\UnaryInfC{$a : A|B, w : A > B \Rightarrow_w w : A \to B$}
\RightLabel{$\mathsf{L} >_{\mathsf{A}}$}
\BinaryInfC{$a \subseteq a, w : A > B \Rightarrow_w w : A \to B$}
\RightLabel{$\mathsf{N_A}$}
\UnaryInfC{$w : A > B \Rightarrow_w w : A \to B$}
\RightLabel{$\mathsf{R} \to_A$}
\UnaryInfC{$\Rightarrow_w w : A > B \to (A \to B)$}
\end{prooftree}
$$

To derive formula b) in $\mathbf{G3AC_w}$ and $\mathbf{G3AVC_w}$ we need to add rule $\mathsf{C_A}^*$ to both calculi. Observe that in the fully modular sequent calculi $\mathbf{G3CL}^*$ it wasn't necessary to add any special rule - b) can be derived using the rules for absoluteness and centering.

$$
\begin{prooftree}
\AxiomC{$y \in \{w\}, a \subseteq \{x\}, y \overset{\triangledown}{\in} a, y : A, \cdots \Rightarrow_w y : A$}
\RightLabel{$\mathsf{R} \Vdash^{\exists}_{\mathsf{A}}$}
\UnaryInfC{$y \in \{w\}, a \subseteq \{x\}, y \in a, y : A, \cdots \Rightarrow_w a \Vdash^{\exists} A$}
\RightLabel{$\mathsf{L} \subseteq_{\mathsf{A}}$}
\UnaryInfC{$a \subseteq \{x\}, y \in a, y : A, \cdots \Rightarrow_w a \Vdash^{\exists} A$}
\RightLabel{$\mathsf{L} \Vdash^{\exists}_{\mathsf{A}}$}
\UnaryInfC{$a \subseteq \{x\}, a \subseteq \{x\}, a \Vdash^{\exists} A, w : A \to B \Rightarrow_w a \Vdash^{\exists} A$}
\RightLabel{$\mathsf{C_A}^*$}
\UnaryInfC{$\{x\} \subseteq a, a \Vdash^{\exists} A \cdots \Rightarrow_w a \Vdash^{\exists} A$}
\AxiomC{$x \in \{w\} \ldots x : A \overset{\triangledown}{\to} B \Rightarrow_w x : A \to B$}
\RightLabel{$\mathsf{Repl_{A2}}$}
\UnaryInfC{$x \in \{w\} \ldots w : A \to B \Rightarrow_w x : A \to B$}
\RightLabel{$\mathsf{R} \Vdash^{\forall}_{\mathsf{A}}$}
\UnaryInfC{$\{w\} \subseteq a \ldots w : A \to B \Rightarrow_w \{w\} \Vdash^{\forall} A \to B$}
\RightLabel{$\mathsf{R}|_{\mathsf{A}}$}
\BinaryInfC{$\{x\} \subseteq a, a \Vdash^{\exists} A, w : A \to B \Rightarrow_w a : A|B$}
\RightLabel{$\mathsf{C_A}$}
\UnaryInfC{$a \Vdash^{\exists} A, w : A \to B \Rightarrow_w a : A|B$}
\RightLabel{$\mathsf{R} >_{\mathsf{A}}$}
\UnaryInfC{$w : A \to B \Rightarrow_w w : A > B$}
\RightLabel{$\mathsf{R} \to_A$}
\UnaryInfC{$\Rightarrow_w w : (A \to B) \to A > B$}
\end{prooftree}
$$

To prove soundness of the rules of $\mathbf{G3A^*_w}$ we need - as usual - to define the notion of realization. This is done as in Definition 3.3.3, with the following modifications, for w world label placed at the root of the derivation tree:

- For all $\sigma(a)$, it holds that $\sigma(a) \in N(\rho(w))$;
- $\mathcal{M} \vDash_{\rho,\sigma} a : A|B$ if for some $\beta \subseteq \sigma(a)$ it holds that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$;
- $\mathcal{M} \vDash_{\rho,\sigma} w : A > B$ if for all $\sigma(a)$ it holds that if $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ then $\mathcal{M} \vDash_{\rho,\sigma} a : A|B$.

Theorem 3.8.1 (Soundness)**.** The rules of $\mathbf{G3A^*_w}$ are sound.

Proof. By induction on the height of the derivation; by means of example, we show the case of RA (case LA is symmetric). Suppose the premiss of RA is valid, whereas the conclusion is not. Thus, there is a model $\mathcal{M}$ and a world x that, under the realization (ρ, σ), does not satisfy the conclusion. Let $\mathcal{M}, x \nvDash_{\rho,\sigma} \Gamma \Rightarrow_w \Delta, w : A > B$, from which we obtain:

1. $\mathcal{M}, x \vDash_{\rho,\sigma} F$ for all $F \in \Gamma$;
2. $\mathcal{M}, x \nvDash_{\rho,\sigma} G$ for all $G \in \Delta$;
3. $\mathcal{M}, x \nvDash_{\rho,\sigma} x : A > B$, i.e., there exists a $\sigma(a)$ such that $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ and $\mathcal{M} \nvDash_{\rho,\sigma} a : A|B$.

By the definition of realization it holds that $\sigma(a) \in N(\rho(w))$. Since the premiss is valid, we have $\mathcal{M}, x \vDash_{\rho,\sigma} \Gamma \Rightarrow_w \Delta, w : A > B$. From 1 and 2, it holds that $\mathcal{M}, x \vDash_{\rho,\sigma} w : A > B$, i.e., for all $\sigma(a)$ it holds that if $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ then $\mathcal{M} \vDash_{\rho,\sigma} a : A|B$. This contradicts 3. □

The desirable structural properties of admissibility of weakening, invertibility of all the rules, admissibility of contraction and admissibility of cut hold for $\mathbf{G3A^*_w}$. Derivability of generalized initial sequents and of generalized versions of the rules of replacement can be proved as for sequent calculus **G3G3CL**. Admissibility of weakening is proved by straightforward induction on the height of the derivation.

Lemma 3.8.2. The rules of weakening are height-preserving admissible in $\mathbf{G3A^*_W}$.

Lemma 3.8.3. All the rules of $\mathbf{G3A^*_w}$ are height-preserving invertible.

Proof. The proof is by induction on the height of the derivation; all cases are straightforwardly obtained by weakening, including the cases of LA and RA. By means of example, let us consider RA. We have to prove that if sequent $\Gamma \Rightarrow_w \Delta, x : A > B$ is derivable, then sequent $\Gamma \Rightarrow_w \Delta, w : A > B$ is derivable with a derivation of the same height. Suppose that $\Gamma \Rightarrow_w \Delta, x : A > B$ is derivable. The sequent cannot be derivable by R $>$, since the rule is applicable only to formulas $w : A > B$. If the sequent has been derived by RA, the lemma is trivially proved. If it has been derived by an initial sequent, the sequent $\Gamma \Rightarrow_w \Delta, y : A > B$ is derivable by weakening from the initial sequent. Finally, if it has been derived by some rule R other than RA, the conclusion is derivable by application of the inductive hypothesis on the premiss of R and application of R. □

Lemma 3.8.4. The rules of contraction are admissible in $\mathbf{G3A^*_W}$.

Proof. Standard: by induction on the height of the derivation, making an essential use of the invertibility result. □

Theorem 3.8.5. The rule of cut is admissible in $\mathbf{G3A^*_W}$.

Proof. The proof is by primary induction on the complexity of the cut formula, and by secondary induction on the sum of the heights of the derivations of the premisses. The case distinction is the same as in Lemma 2.2.4. The only cases that differ are those involving rules RA and LA, which do not present particular difficulties.

- The cut formula is not the principal in at least one of the premisses of cut, and the other premiss is derived by LA (case of RA is similar):

$$\dfrac{\dfrac{\overset{(1)}{\Gamma^* \Rightarrow_w \Delta^*, x : A > B}}{\Gamma \Rightarrow_w \Delta, x : A > B}\, R \quad \dfrac{w : A > B, \Gamma' \Rightarrow_w \Delta'}{x : A > B, \Gamma' \Rightarrow_w \Delta'}\, \mathsf{LA}}{\Gamma, \Gamma' \Rightarrow_w \Delta, \Delta'}\, \mathsf{cut}$$

Apply a cut on smaller derivation height on (1) and $x : A > B, \Gamma' \Rightarrow_w \Delta'$; then apply R.

- The cut formula is principal in both premisses and it is derived by RA and LA respectively:

$$\dfrac{\dfrac{\overset{(1)}{\Gamma \Rightarrow_w \Delta, w : A > B}}{\Gamma \Rightarrow_w \Delta, x : A > B}\,\mathsf{RA} \quad \dfrac{\overset{(2)}{w : A > B, \Gamma' \Rightarrow_w \Delta'}}{x : A > B, \Gamma' \Rightarrow_w \Delta'}\,\mathsf{LA}}{\Gamma, \Gamma' \Rightarrow_w \Delta, \Delta'}\,\mathsf{cut}$$

Apply cut to (1) and (2): the application has smaller sum of derivation height than the original application. □

Theorem 3.8.6. The axioms and inference rules of $\mathbb{PCLA}$ and its extensions are derivable in $\mathbf{G3A^*_w}$.

Proof. Refer to Appendix 3.A at the end of this chapter. □

To prove termination of the calculi $\mathbf{G3A^*_w}$ we proceed as in the previous sections. We start by defining saturation conditions associated to each rule, and a proof search strategy to be applied in root-first proof search. Then, we prove that the graph obtained from the labels occurring in any derivation is finite: each node has a finite number of immediate successors and each branch has a finite length.

Definition 3.8.2. Let $\mathcal{B} = S_0, S_1, \ldots$ with S_i sequent $\Gamma_i \Rightarrow \Delta_i$ and S_0 sequent $\Rightarrow_w w : A_0$. The saturation conditions associated to each rule are slight modifications of the condition stated in Definition 3.5.2. We detail the most relevant ones.

($\mathsf{R} >_{\mathsf{A}}$) If $x : A > B$ is in $\downarrow \Delta$, then for some a, $a \Vdash^{\exists} A$ is in $\downarrow \Gamma$ and $a : B|A$ is in Δ;
($\mathsf{L} >_{\mathsf{A}}$) If $x : A > B$ is in Γ and the neighbourhood label a occurs in $\downarrow \Gamma$, then either $a \Vdash^{\exists} A$ is in Δ or $a : B|A$ are in $\downarrow \Gamma$;
($\mathsf{R}|_{\mathsf{A}}$) If $c \subseteq a$ are in Γ and $a : B|A$ is in Δ, then either $c \Vdash^{\exists} A$ is in Δ or $c \Vdash^{\forall} A \to B$ is in $\downarrow \Delta$;
($\mathsf{L}|_{\mathsf{A}}$) If $a : B|A$ is in $\downarrow \Gamma$, then for some c, $c \subseteq a$ is in Γ, $c \Vdash^{\exists} A$ is in $\downarrow \Gamma$ and $c \Vdash^{\forall} A \to B$ is in Γ.
(0_{A}) If a occurs in Γ and some $a \Vdash^{\exists} A$ or $a \Vdash^{\forall} A$ occurs in $\downarrow \Gamma \cup \downarrow \Delta$ then $y \in a$ occurs in Γ, for some a;
(N_{A}) $a \subseteq a$ occurs in Γ, for some a;
(T_{A}) There is an a such that $w \in a$ is in Γ;
(W_{A}) If a occurs in Γ then $w \in a$ occurs in Γ;
(C_{A}) If x and a occur in Γ, $\{x\} \subseteq a$ occurs in Γ;
($\mathsf{Single}_{\mathsf{A}}$) If $\{x\}$ occurs in Γ, then $x \in \{x\}$ occurs in Γ;
($\mathsf{Repl}_{\mathsf{A1}}$) If $y \in \{x\}$ occurs in Γ, and if some formula $At(x)$ occurs in Γ, then $At(y)$ occurs in Γ;
($\mathsf{Repl}_{\mathsf{A2}}$) If $y \in \{x\}$ occurs in Γ, and if some formula $At(y)$ occurs in Γ, then $At(x)$ occurs in Γ;
($\mathsf{Nes}_{\mathsf{A}}$) If a and b occur in $\downarrow \Gamma$, then either $a \subseteq b$ or $b \subseteq a$ occur in Γ.

Definition 3.8.3. The proof search strategy consists of the same clauses (i) - (v) from Definition 3.5.3, to which we add:

(vi) Rule LA is applied *before* any other rule.

Definition 3.8.4. Given a derivation branch, let a, b be neighbourhood labels and x, y world labels occurring in $\downarrow \Gamma_k$.

- $k(x) = min\{t \mid x \ occurs \ in \ \Gamma_t\}$;

- $k(a) = min\{t \mid a \ occurs \ in \ \Gamma_t\}$;
- $x \to_g a$, "x generates a" if, given $k(a) = t$, for some $t \leqslant k$, a occurs in Γ_t;
- $b \to_g y$, "b generates y" if, given $k(y) = t$, for some $t \leqslant k$, $y \in b$ occurs in Γ_t;
- $x \xrightarrow{w} y$ if for some a, $x \to_g a$ and $a \to_g y$ and $x \neq y$.

Given a branch $\mathcal{B}$ as in Definition 3.8.2, and given a world label x, let $\mathsf{Neigh}(x) = \{a \mid x \to_g a\}$ and $\mathsf{World}(x) = \{y \mid x \xrightarrow{w} y\}$.

As for the termination proofs in the previous sections, we have that the relation $\xrightarrow{w}$ forms an acyclic graph with label w at the root, and that all the labels occurring in a derivation branch occur in the graph.

Proposition 3.8.7. The maximal length of each branch of the graph of labels associated to a $\mathbf{G3A^*_w}$ derivation branch starting from sequent $\Rightarrow w : A_0$ is at most one.

Proof. If the length of each branch is at most one, it means that the graph of labels associated to a derivation is composed only of labels y such that $w \xrightarrow{w} y$. Suppose that $w \xrightarrow{w} y$. We show that there are no z such that $y \to_g z$, i.e., that there is no neighbourhood label b such that $y \to_g b$ and $b \to_g z$. The only way in which y could generate new neighbourhood label is by application of one of the rules for the extensions, more precisely $\mathsf{N_A}$, $\mathsf{T_A}$ or $\mathsf{C_A}$, or by application of $\mathsf{R} >_\mathsf{A}$ or $\mathsf{L}|_\mathsf{A}$. However, by definition of $\mathbf{G3A^*_w}$ sequents, all neighbourhood labels introduced in the derivation belong to $N(w)$. Thus, y cannot generate new neighbourhood labels, and consequently no new world label. Moreover, observe that if some formula $y : C > D$ is introduced in the derivation, the proof-search strategy imposes to apply the rules for absoluteness, which "report" the conditional formula to the world at the root of the tree of labels, w. Thus, all neighbourhood labels occurring in the derivation belong to $N(w)$, and all worlds labels y occurring in the derivation are such that $w \xrightarrow{w} y$. $\square$

The following lemma proves that every node of the graph of labels associated to a branch of derivations has a finite number of immediate successors. By the previous Proposition, the only world that can generate other world labels is w. Thus, we count the maximal number of neighbourhood and world labels which might be generated from w. These will be also the maximal number of neighbourhood and world labels that can occur in the branch.

Lemma 3.8.8. Given a branch $\mathcal{B}$ as in Definition 3.8.2, the size of $\mathsf{Neigh}(w)$ and $\mathsf{World}(w)$ is finite.

Proof. We distinguish cases according to the conditional degree of the formula A_0 placed at the root of the derivation tree. If $d(A_0) = 0$, the formula is composed only of propositional connectives. In case of sequent calculus $\mathbf{G3A_w}$, no neighbourhood or world labels are introduced. In case of sequent calculi for extensions, at most three neighbourhood labels can be introduced, by application of $\mathsf{N_A}$, $\mathsf{T_A}$ and $\mathsf{C_A}$, this latter introducing neighbourhood $\{w\}$. Then, rule $\mathsf{0_A}$ might introduce at most one world label.

If $d(A_0) = 1$, A_0 contains some subformula $A > B$. To count neighbourhood labels generated from w in this case, the strategy is the same as in proof of Lemma 3.5.4. By definition, $w \to_g a$ if there exists a t such that a does not occur in Γ_s for all $s < t$ and a occurs in Γ_t. Thus, label a has been introduced in the derivation either by some rule for semantic conditions, by $\mathsf{R} >_\mathsf{A}$ or $\mathsf{L}|_\mathsf{A}$. Again, each rules for semantic conditions $\mathsf{N_A}$, $\mathsf{T_A}$ and $\mathsf{C_A}$ can be applied at most once, introducing at most three neighbourhood labels.

In case of $\mathsf{R} >_\mathsf{A}$, one neighbourhood label can be generated by some formula $x : C > D$ occurring in $\downarrow \Delta_s$. The number of such formulas in Δ_s depends on the conditional degree of A_0. Let $n = d(A_0)$; thus, at most $O(n)$ neighbourhood labels are introduced.

The case in which $a \in N(w)$ is generated by application of $\mathsf{L}|_\mathsf{A}$ is far more complex: as in **G3CL**, a loop might arise from the interaction of $\mathsf{L}|_\mathsf{A}$ with $\mathsf{L} >_\mathsf{A}$. To generate a finite number of neighbourhood labels, we apply the same solution as for **G3CL**. We introduce a couple of new (admissible) rules $\mathsf{Mon}\forall_\mathsf{A}$ and $\mathsf{L} >_\mathsf{A}{}'$, and obtain that, thanks to the saturation condition associated to these rules, the number of neighbourhood labels generated in this case is *finite*. We do not detail the proof, similar to the one in proof of Lemma 3.5.4. The number of neighbourhood labels generated in this case is $O(n!)$.

$$\frac{w : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad a \Vdash^{\exists} A, a : A|B, w : A > B, \Gamma \Rightarrow \Delta}{w : A > B, \Gamma \Rightarrow \Delta}\ \mathsf{L} >_\mathsf{A}{}'$$

$$\frac{b \subseteq a, b \Vdash^{\forall} A, a \Vdash^{\forall} A, \Gamma \Rightarrow_w \Delta}{b \subseteq a, a \Vdash^{\forall} A, \Gamma \Rightarrow_w \Delta}\ \mathsf{Mon}\forall_\mathsf{A}$$

According to the proof-search strategy, rules $\mathsf{N_A}$, $\mathsf{T_A}$, $\mathsf{C_A}$ and $\mathsf{R} >_\mathsf{A}$ are applied before $\mathsf{L} >_\mathsf{A}$ and $\mathsf{L}|_\mathsf{A}{}'$, and they produce at most $O(n+3) = O(n)$ neighbourhood labels. To these labels are applied rules $\mathsf{L} >_\mathsf{A}$ and $\mathsf{L}|_\mathsf{A}{}'$; thus, the number of neighbourhood labels that can be introduced from w is $|\mathsf{Neigh}(w)| = O(n \cdot n!)$.

By definition, $y \in W(w)$ if $w \xrightarrow{w} y$, i.e. if for some b it holds that $w \rightarrow_g b$ and $b \rightarrow_g y$. Let us consider $b \rightarrow_g y$. This means that there exists t such that y does not occur in Γ_s for $s < t$ and $y \in b$ occurs in Γ_t. To count the number of formulas $y \in b$, we have to consider how the formula might be introduced. The case distinction is the same as in proof of Lemma 3.5.4, and the result is the same: each $b \in \mathsf{Neigh}(w)$ generates at most $O(n)$ world labels. We have shown that $|\mathsf{Neigh}(w)| = O(n \cdot n!)$. The number of world labels that can be generated from a world label x is given by $|\mathsf{World}(w)| = O(n^2 \cdot n!)$.

If $d(A_0) > 1$, the count above does not suffice. At some point a formula $y : C > D$ will be introduced in the derivation branch and, following the proof-search strategy, either LA or RA is applied, obtaining $w : C > D$. Formula $C > D$ is a subformula of A_0; moreover, it has a conditional degree strictly smaller than A_0, since it has been obtained by a combination of conditional rules (the argument is similar to the one in proof of Lemma 3.5.5). If $d(w : C > D) = 1$, the formula generates $O(n \cdot n!)$ neighbourhood and $O(n^2 \cdot n!)$ world labels, to be added to the count. If $d(w : C > D) > 1$, a formula $z : C$ will be introduced in the derivation, with $w \xrightarrow{w} z$ and some conditional formula $E > F$ subformula of C. Again, we apply the rules for absoluteness and add to the count the labels that this latter formula might generate.

Thus, the total number of neighbourhood labels which might be introduced in the derivation is at most $|\mathsf{Neigh}(w)| = O(n \cdot n! \cdot n) = O(n^2 \cdot n!)$; and the total number of world labels is at most $|\mathsf{World}(w)| = O(n^2 \cdot n! \cdot n) = O(n^3 \cdot n!)$. Observe that the number of neighbourhood labels does not depend on the world labels that might introduced, since the rules of absoluteness "report" all conditional formulas to the same world w. The number of neighbourhood depends instead on the level of nesting of the conditional formula at the root. $\square$

Remark 3.8.2. To obtain a smaller bound on the number of labels that can be introduced in a derivation, we can modify the calculus $\mathbf{G3AN_w}$ (with nesting on neighbourhood) in a similar way as done in the section for nesting, i.e., by defining rules for the comparative plausibility operator.

$$\frac{a \Vdash^{\exists} B, \Gamma \Rightarrow_w \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow_w \Delta, w : A \preccurlyeq B}\ \mathsf{R} \preccurlyeq_A (\mathsf{a!})$$

$$\frac{x : A \preccurlyeq B, \Gamma \Rightarrow_w \Delta, a \Vdash^{\exists} B \quad a \Vdash^{\exists} A, x : A \preccurlyeq B, \Gamma \Rightarrow_w \Delta}{w : A \preccurlyeq B, \Gamma \Rightarrow \Delta} \; \mathsf{L} \preccurlyeq_A$$

With this operation, we have that $|\mathsf{Neigh}(w) = O(n^2)$, and $|\mathsf{World}(w)| = O(n^3)$.

Corollary 3.8.9 (Termination)**.** Proof search in $\mathbf{G3A_w^*}$ comes to an end in a finite number of steps.

Proof. We start derivations with sequent $\Rightarrow_w w : A_0$. Applying rules according to the proof-search strategy we generate only a *finite* number of subformulas of A_0. Thus, the only way for a derivation branch to be infinite is for the labels occurring in it to be infinite. To check this, we build an acyclic graph with all the world labels that might possibly occur in a derivation, as described in Definition 3.8.4. By Lemma 3.8, the maximal number of world labels that can be generated from w is finite. Moreover, by Proposition 3.8.7, each path in the graph has at most length one. This suffices to prove that the graph of labels is finite: every node has a finite number of immediate successors, and each path is of finite length. Thus, there are no infinite labels, and root-fists proof search comes to an end in a finite number of steps. □

It is possible to prove semantic completeness for the calculi $\mathbf{G3A_w^*}$, using a similar construction as the one employed in Section 3.6. We first show completeness for systems without strong centering, since the condition requires a different countermodel construction.

Theorem 3.8.10. Let $\Gamma \Rightarrow_w \Delta$ be a saturated upper sequent in a derivation in $\mathbf{G3A_w^*}$. Then there exists a *finite* countermodel $\mathcal{M}$ that satisfies all formulas in $\downarrow \Gamma$ and falsifies all formulas in $\downarrow \Delta$.

Proof. We start by showing the countermodel construction in case of systems $\mathbf{G3A_w}$, $\mathbf{G3NA_w}$, $\mathbf{G3TA_w}$, $\mathbf{G3VA_w}$, $\mathbf{G3VNA_w}$ or $\mathbf{G3VTA_w}$. The cases of $\mathbf{G3CA_w}$ and $\mathbf{G3VCA_w}$ require a different construction, since these calculi introduce neighbourhoods of the form $\{x\}$.

Let $\Gamma \Rightarrow_w \Delta$ be a saturated upper sequent of a branch $\mathcal{B}$. Let

$$S_{\mathcal{B}} = \{x \mid x \in (\downarrow \Gamma \cup \downarrow \Delta)\} \qquad N_{\mathcal{B}} = \{a \mid a \in (\downarrow \Gamma \cup \downarrow \Delta)\}$$

Then, for each $a \in N_{\mathcal{B}}$, define a neighbourhood

$$\alpha_a = \{y \in S_{\mathcal{B}} \mid y \in a \textit{ belongs to } \Gamma\}$$

We construct a neighbourhood model $\mathcal{M}_A = \langle W_A, N_A, [\![\ ]\!]_A \rangle$ as follows.

- $W_A = S_{\mathcal{B}}$
- $N_A = \{\alpha_a \mid a \textit{ belongs to } \downarrow \Gamma\}$
- For p atomic, $[\![p]\!]_A = \{x \in W_{\mathcal{B}} \mid x : P \textit{ belongs to } \downarrow \Gamma\}$

The difference with the countermodel construction described in Section 3.6 is the definition of the system of neighbourhoods N_A: here, to each world we associate the same system of neighbourhood, composed of the neighbourhood labels occurring in the branch. It is immediate to verify that such a model satisfies the properties of local absoluteness. As for the properties of non-emptiness, normality, total reflexivity, weak centering and nesting, these are verified as in proof of Theorem 3.6.1 and Theorem 3.7.9, making reference to the saturation condition associated to each rule.

To complete the countermodel construction, we define a realization (ρ, σ) such that $\rho(x) = x$ and $\sigma(a) = \alpha_a$, and prove the following claims:

[**Claim 1**] If $\mathcal{F}$ is in $\downarrow \Gamma$, then $\mathcal{M}_{\mathcal{B}} \vDash \mathcal{F}$;
[**Claim 2**] If $\mathcal{F}$ is in $\downarrow \Delta$, then $\mathcal{M}_{\mathcal{B}} \nvDash \mathcal{F}$;

The two claims are proved by cases, by induction on the weight of the formula $\mathcal{F}$. We show only the case of $\mathcal{F} = x : A > B$.
Suppose $x : A > B$ is in $\downarrow \Gamma$. Then, by the saturation condition associated to LA, formula $w : A > B$ occurs in Γ. Consider an arbitrary neighbourhood α_a. By definition of $\mathcal{M}_A$ it holds that a occurs in Γ; apply the saturation condition $(\mathsf{L} >_{\mathsf{A}})$ and conclude that either $a \Vdash^{\exists} A$ is in $\downarrow \Delta$, or $a : A|B$ is in $\downarrow \Gamma$. By inductive hypothesis, it holds that either $\mathcal{M}_A \nvDash \alpha_a \Vdash^{\exists} A$ or $\mathcal{M}_A \vDash a : A|B$. In both cases, by definition $\mathcal{M}_A \vDash x : A > B$.
If $x : A > B$ is in $\downarrow \Delta$, by the saturation condition (RA) formula $w : A > B$ occurs in $\downarrow \Delta$. By the saturation condition $(\mathsf{R} >_{\mathsf{A}})$, for some a it holds that for some a, $a \Vdash^{\exists} A$ occurs in $\downarrow \Gamma$ and $a : A|B$ is in $\downarrow \Delta$. By inductive hypothesis, $\mathcal{M}_A \vDash \alpha_a \Vdash^{\exists} A$ and $\mathcal{M}_A \nvDash a : A|B$, thus, by definition, we have $\mathcal{M}_A \nvDash x : A > B$.

To extend the theorem to sequent calculi $\mathbf{G3CA_w}$ and $\mathbf{G3VCA_w}$ we adopt the countermodel construction described in the proof of Theorem 3.6.1 for the case of centering, with the modifications applied in the proof of the previous cases. Thus, the worlds of a countermodel $\mathcal{M}_A^c$ for systems with strong centering are taken to be equivalence classes $[x]$ with respect to the relation $y \in x$. For $S_{\mathcal{B}}, N_{\mathcal{B}}$ and α_a defined as before, let:

$[x] = \{y \in S_{\mathcal{B}} \mid y \in \{x\} \textit{ occurs in } \Gamma\}$;
$[x] \subseteq \alpha_a$, for a occurring in Γ.

A countermodel $\mathcal{M}_A^c = \langle W_A^c, N_A^c, [\![\ ]\!]_A^c \rangle$ is composed of the following elements:

- $W_A^c = \{[x] \mid x \in S_{\mathcal{B}}\}$;
- $N_A^c = \{\alpha_a \mid a \textit{ belongs to } \downarrow \Gamma\}$;
- For p atomic, $[\![p]\!]_A^c = \{[x] \in W^c \mid x : P \textit{ belongs to } \downarrow \Gamma\}$.

It is immediate to verify that such a model satisfies the conditions of centering. Moreover, observe that from the saturation condition associated to $\mathsf{C_A}^*$ it holds that $\alpha \subseteq [x]$. This corresponds to the requirement that models for $\mathbb{PCLCA}$ and $\mathbb{VCA}$ should be composed of just one sphere containing only the actual world since these logics collapse to classical logic (Remark 3.8.1). The rest of the proof consists in showing that $y \in \{x\}$ is an equivalence relation, that the definition of $[\![p]\!]_A^c$ does not depend on the chosen representative of the equivalence class and, finally, that Claim 1 and Claim 2 hold. We do not detail the proof, which is the same as the proof of Theorem 3.6.1. $\square$

3.9 To conclude

In the first half of this chapter we have presented a family of labelled sequent calculi $\mathbf{G3CL}^*$ for conditional logic $\mathbb{PCL}$ and its extensions. The calculi are based on neighbourhood semantics, which is a generalization of sphere semantics allowing to capture a large number of conditional logics. The family $\mathbf{G3CL}^*$ is fully modular, meaning that it is possible to define a sequent calculus for logics extending $\mathbb{PCL}$ by adding rules in correspondence with semantic conditions.

In the second half of the chapter we have introduced slightly different proof systems, better suited to formalize logics with certain semantic conditions. Thus, we have defined calculi $\mathbf{G3V}^*$, for conditional logic with nesting, corresponding to Lewis' counterfactual logics. Calculi $\mathbf{G3V}^*$ display a smaller number of formulas in each sequent, and have a smaller number of rules. The sequent calculus $\mathbf{G3V}$ is used again in Chapter 5, in the scope of an equivalence result between proof systems.

We have then defined a family of sequent calculi $\mathbf{G3A}^*_{\mathbf{w}}$ for logics with the condition of absoluteness, exploiting the fact that models for these logics associate to each world the same system of neighbourhoods. Interestingly enough, these proof systems display modifications on the structure of sequents: one neighbourhood label is generalized throughout the model, and used to analyse all labelled formulas.

We showed that all the families of sequent calculi considered enjoy cut admissibility, and proved termination and semantic completeness for the majority of the systems. We have not proved termination for logics with uniformity: for these systems, completeness is proved only syntactically (via derivation of the axioms). As a general remark, the calculi have strong proof-theoretical properties; however, it can be observed that the proofs of termination require a long and complex procedure (especially in the non-nested case) and that the bound on the maximal number of labelled formulas which could be introduced in the derivation is quite high. Thus, the proofs of formulas using the labelled systems introduced in this chapter are quite long, and the resulting decision procedure for the logics is far from optimal.

One could wonder whether it is possible to define internal sequent calculi for conditional logics, and which properties these calculi would have. To the best of our knowledge, other than the resolution calculus in [68], no standard and internal proof system for $\mathbb{PCL}$ and its non-nested extensions has been found. In this direction, next chapter presents internal and standard calculi for counterfactual logic $\mathbb{V}$ and its extensions.

3.A Appendix: derivations of the axioms

Proof of Theorem 3.4.7. To prove completeness of $\mathbf{G3CL}^*$ with respect to the axiomatization, we show that the inference rules of $\mathbb{PCL}$ are admissible in $\mathbf{G3CL}^*$, and that the axioms of $\mathbb{PCL}$ and extensions are derivable in the corresponding proof systems. For reasons of space, not all sequents of the derivation are written in full; some formula are replaced by dots. The rules for negation can be defined from the rules for implication.

$$\frac{\Gamma \Rightarrow \Delta, x : A}{x : \neg A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\neg \qquad\qquad \frac{x : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : \neg A}\ \mathsf{R}\neg$$

We recall the axioms and inference rules of $\mathbb{PCL}$ and its extensions, from Chapter 1.

$\mathbb{PCL}$		Axiomatization of classical propositional logic
	(RCEA)	$\dfrac{A \leftrightarrow B}{(A > C) \leftrightarrow (B > C)}$
	(RCK)	$\dfrac{A \to B}{(C > A) \to (C > B)}$
	(R-And)	$(A > B) \wedge (A > C) \to (A > (B \wedge C))$
	(ID)	$A > A$
	(CM)	$(A > B) \wedge (A > C) \to ((A \wedge B) > C)$
	(RT)	$(A > B) \wedge ((A \wedge B) > C) \to (A > C)$
	(OR)	$(A > C) \wedge (B > C) \to ((A \vee B) > C)$
Extensions	(N)	$\neg(\top > \bot)$
	(T)	$A \to \neg(A > \bot)$
	(W)	$(A > B) \to (A \to B)$
	(C)	$(A \wedge B) \to (A > B)$
	(U_1)	$(\neg A > \bot) \to \neg(\neg A > \bot) > \bot$
	(U_2)	$\neg(A > \bot) \to ((A > \bot) > \bot)$
	(A_1)	$(A > B) \to (C > (A > B))$
	(A_2)	$\neg(A > B) \to (C > \neg(A > B))$
	(CV)	$((A > C) \wedge \neg(A > \neg B)) \to ((A \wedge B) > C)$

To prove completeness of **G3CL*** with respect to axioms and inference rules of classical propositional logic we use only propositional rules; the proof is standard, and to show admissibility of *modus ponens* we need admissibility of cut (Theorem 3.4.6).

For (RCEA), suppose $\vdash A \leftrightarrow B$. Thus, $\vdash x : A \to B$, whence $\Rightarrow x : A \to B$, and $\vdash x : B \to A$, whence $\Rightarrow x : B \to A$. We derive three sequents by application of cut and other rules, and show how to combine them into a derivation of $\Rightarrow x : (A > C) \to (B > C)$, i.e., $\vdash (A > C) \to (B > C)$. The other direction of the implication is similar.

From $\Rightarrow x : B \to A$ obtain by substitution $\Rightarrow y : B \to A$. Sequent $y : B, y : B \to A \Rightarrow y : A$ is derivable.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow y : B \to A,}{y : A \Rightarrow y : B \to A, y : B}\,\mathsf{Wk} \qquad y : B, y : B \to A \Rightarrow y : A}{y : A \Rightarrow y : B}\,\mathsf{cut}}{a \in N(x), y \in a, y : B, x : A > C \Rightarrow x \Vdash_a B|C, a \Vdash^{\exists} A, y : A}\,\mathsf{Wk}}{a \in N(x), y \in a, y : B, x : A > C \Rightarrow x \Vdash_a B|C, a \Vdash^{\exists} A}\,\mathsf{R}\Vdash^{\exists}}{(1)\quad a \in N(x), a \Vdash^{\exists} B, x : A > C \Rightarrow x \Vdash_a B|C, a \Vdash^{\exists} A}\,\mathsf{L}\Vdash^{\exists}$$

From $\Rightarrow x : A \to B$ obtain by substitution $\Rightarrow y : A \to B$. Sequent $y : A, y : A \to B \Rightarrow y : B$ is derivable.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow y : A \to B}{y : A \Rightarrow y : A \to B, y : B}\,\mathsf{Wk} \qquad y : A, y : A \to B \Rightarrow y : B}{y : A \Rightarrow y : B}\,\mathsf{cut}}{c \subseteq a, a \in N(x), c \in N(x), y \in c, a \Vdash^{\exists} B, y : A, c \Vdash^{\forall} A \to C, y : A \to C, , x : A > C \Rightarrow x \Vdash_a B|C}\,\mathsf{Wk}}{c \subseteq a, a \in N(x), c \in N(x), y \in c, a \Vdash^{\exists} B, y : A, c \Vdash^{\forall} A \to C, , x : A > C \Rightarrow x \Vdash_a B|C}\,\mathsf{R}\Vdash^{\exists}}{(2)\quad c \subseteq a, a \in N(x), c \in N(x), a \Vdash^{\exists} B, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, x : A > C \Rightarrow x \Vdash_a B|C}\,\mathsf{L}\Vdash^{\exists}$$

From $\Rightarrow x : B \to A$ obtain by substitution $\Rightarrow z : B \to A$, for some variable z different from y. Sequent $z : C \Rightarrow z : C$ is derivable, as well as $z : B, z : B \to A \Rightarrow z : A$.

$$
\dfrac{\dfrac{\dfrac{z : C \Rightarrow z : C \qquad \dfrac{\dfrac{\Rightarrow z : B \to A}{z : B \Rightarrow z : B \to A, z : A}\,\mathsf{Wk} \qquad z : B, z : B \to A \Rightarrow z : A}{z : B \Rightarrow z : A}\,\mathsf{cut}}{\dfrac{z : A \to C, z : B \Rightarrow z : C}{z : A \to C \Rightarrow z : B \to C}\,\mathsf{R}\to}\,\mathsf{L}\to}{z \in c, c \subseteq a, a \in N(x), c \in N(x), c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, z : A \to C, a \Vdash^{\exists} B, x : A > C \Rightarrow x \Vdash_a B|C, z : B \to C}\,\mathsf{Wk}}{\dfrac{z \in c, c \subseteq a, a \in N(x), c \in N(x), c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, a \Vdash^{\exists} B, x : A > C \Rightarrow x \Vdash_a B|C, z : B \to C}{(3)\ c \subseteq a, a \in N(x), c \in N(x), c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, a \Vdash^{\exists} B, x : A > C \Rightarrow x \Vdash_a B|C, c \Vdash^{\forall} B \to C}\,\mathsf{R}\Vdash^{\forall}}\,\mathsf{L}\Vdash^{\forall}
$$

To conclude, we combine the above derivations into the following:

$$
\dfrac{\dfrac{\dfrac{(1) \qquad \dfrac{\dfrac{(2) \qquad (3)}{c \subseteq a, a \in N(x), c \in N(x), c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, a \Vdash^{\exists} B, x : A > C \Rightarrow x \Vdash_a B|C}\,\mathsf{R}|}{x \Vdash_a A|C, a \Vdash^{\exists} B, x : A > C \Rightarrow x \Vdash_a B|C}\,\mathsf{L}|}{a \in N(x), a \Vdash^{\exists} B, x : A > C \Rightarrow x \Vdash_a B|C}\,\mathsf{L}>}{x : A > C \Rightarrow x : B > C}\,\mathsf{R}>}{\Rightarrow x : (A > C) \to (B > C)}\,\mathsf{R}\to
$$

For (RCK), suppose $\vdash A \to B$. Thus, $\Rightarrow x : A \to B$. We show how to derive sequent $\Rightarrow x : (C > A) \to (C > B)$. From $\Rightarrow x : A \to B$ obtain by substitution $\Rightarrow y : A \to B$. Then, from this sequent and the derivable sequent $y : C \Rightarrow y : C$ obtain sequent (1):

$$
\dfrac{\dfrac{y : C \to A \Rightarrow y : C \to B \qquad \dfrac{\dfrac{y : C \Rightarrow y : C \qquad \dfrac{y : A \Rightarrow y : B}{y : C, y : A \Rightarrow y : B}\,\mathsf{Wk}}{y : C \to A, y : C \Rightarrow y : B}\,\mathsf{L}\to}{y : C \to A \Rightarrow y : C \to B}\,\mathsf{R}\to}{y \in b, b \subseteq a, a \in N(x), b \in N(x), b \Vdash^{\exists} C, b \Vdash^{\exists} C \to A, y : C \to A, a \Vdash^{\exists} C, x : C > A \Rightarrow x \Vdash_a C|B, y : C \to B}\,\mathsf{Wk}}{\dfrac{y \in b, b \subseteq a, a \in N(x), b \in N(x), b \Vdash^{\exists} C, b \Vdash^{\exists} C \to A, a \Vdash^{\exists} C, x : C > A \Rightarrow x \Vdash_a C|B, y : C \to B}{(1) \quad b \subseteq a, a \in N(x), b \in N(x), b \Vdash^{\exists} C, b \Vdash^{\exists} C \to A, a \Vdash^{\exists} C, x : C > A \Rightarrow x \Vdash_a C|B, b \Vdash^{\forall} C \to B}\,\mathsf{R}\Vdash^{\forall}}\,\mathsf{L}\Vdash^{\forall}
$$

The following two sequents are derivable, by Lemma 3.4.2:

(2) $a \in N(x), a \Vdash^{\exists} C, x : C > A \Rightarrow x \Vdash_a C|B, a \Vdash^{\exists} C$
(3) $b \subseteq a, a \in N(x), b \in N(x), b \Vdash^{\exists} C, b \Vdash^{\exists} C \to A, a \Vdash^{\exists} C, x : C > A \Rightarrow x \Vdash_a C|B, b \Vdash^{\exists} C$

To conclude the proof, apply the following rules to $y : C \to A \Rightarrow y : C \to B$:

$$
\dfrac{\dfrac{\dfrac{(2) \qquad \dfrac{\dfrac{(3) \qquad (1)}{b \subseteq a, a \in N(x), b \in N(x), b \Vdash^{\exists} C, b \Vdash^{\exists} C \to A, a \Vdash^{\exists} C, x : C > A \Rightarrow x \Vdash_a C|B}\,\mathsf{R}|}{x \Vdash_a C|A, a \in N(x), a \Vdash^{\exists} C, x : C > A \Rightarrow x \Vdash_a C|B}\,\mathsf{L}|}{a \in N(x), a \Vdash^{\exists} C, x : C > A \Rightarrow x \Vdash_a C|B}\,\mathsf{L}>}{x : C > A \Rightarrow x : C > B}\,\mathsf{R}>}{\Rightarrow x : (C > A) \to (C > B)}\,\mathsf{R}\to
$$

(R-And)

$$\dfrac{\dfrac{y:B, y:A \cdots \overset{\triangledown}{\Rightarrow} \ldots y:B \quad y:A\cdots \overset{\triangledown}{\Rightarrow} \ldots y:B, y:A}{y:A\to C, y:A\to B, y:A\cdots \Rightarrow \ldots y:B}\ \mathsf{L}\to \qquad \dfrac{y:C, y:A\cdots \overset{\triangledown}{\Rightarrow} \ldots y:C \quad y:A\cdots \overset{\triangledown}{\Rightarrow} \ldots y:C, y:A}{y:A\to C, y:A\to B, y:A\cdots \Rightarrow \ldots y:C}\ \mathsf{L}\to}{y\in b, y\in c, c\subseteq a, c\subseteq b, c\Vdash^{\exists} A, y:A\to C, b\Vdash^{\exists} A, y:A\to B, a\Vdash^{\exists} A, y:A\cdots\Rightarrow \ldots y:B\wedge C}\ \mathsf{L}\wedge$$

$$\dfrac{y\in b, y\in c, c\subseteq a, c\subseteq b, c\Vdash^{\exists} A, y:A\to C, b\Vdash^{\exists} A, y:A\to B, a\Vdash^{\exists} A, y:A\cdots\Rightarrow \ldots y:B\wedge C}{y\in b, y\in c, c\subseteq a, c\in N(x), c\subseteq b, c\Vdash^{\exists} A, y:A\to C, b\Vdash^{\exists} A, b\Vdash^{\forall} A\to B, a\Vdash^{\exists} A, y:A\cdots\Rightarrow \ldots y:B\wedge C}\ \mathsf{L}\Vdash^{\forall}$$

$$\dfrac{}{y\in c, c\subseteq a, c\in N(x), c\subseteq b, c\Vdash^{\exists} A, y:A\to C, b\Vdash^{\exists} A, b\Vdash^{\forall} A\to B, a\Vdash^{\exists} A, y:A\cdots\Rightarrow \ldots y:B\wedge C}\ \mathsf{L}\subseteq$$

$$\dfrac{}{y\in c, c\subseteq a, c\in N(x), c\subseteq b, c\Vdash^{\exists} A, c\Vdash^{\forall} A\to C, b\Vdash^{\exists} A, b\Vdash^{\forall} A\to B, a\Vdash^{\exists} A, y:A\cdots\Rightarrow \ldots y:B\wedge C}\ \mathsf{L}\Vdash^{\forall}$$

$$\dfrac{}{y\in c, c\subseteq a, c\in N(x), c\subseteq b, c\Vdash^{\exists} A, c\Vdash^{\forall} A\to C, b\Vdash^{\exists} A, b\Vdash^{\forall} A\to B, a\Vdash^{\exists} A\cdots\Rightarrow \ldots y:A\to B\wedge C}\ \mathsf{R}\to$$

$$\dfrac{}{c\subseteq a, c\in N(x), c\subseteq b, c\Vdash^{\exists} A, \Vdash^{\forall} A\to C, b\Vdash^{\exists} A, b\Vdash^{\forall} A\to B, a\Vdash^{\exists} A\cdots\Rightarrow \ldots c\Vdash^{\forall} A\to B\wedge C}\ \mathsf{R}\Vdash^{\forall}$$

$$\dfrac{\triangledown \qquad c\subseteq a, c\in N(x), c\subseteq b, c\Vdash^{\exists} A, \Vdash^{\forall} A\to C, b\Vdash^{\exists} A, b\Vdash^{\forall} A\to B, a\Vdash^{\exists} A\cdots\Rightarrow \ldots c\Vdash^{\forall} A\to B\wedge C}{c\subseteq a, c\in N(x), c\subseteq b, c\Vdash^{\exists} A, \Vdash^{\forall} A\to C, b\Vdash^{\exists} A, b\Vdash^{\forall} A\to B, a\Vdash^{\exists} A\cdots\Rightarrow x\Vdash_a A|B\wedge C}\ \mathsf{R}|$$

$$\dfrac{}{c\in N(x), c\subseteq b, b\in N(x), b\subseteq a, a\in N(x), c\Vdash^{\exists} A, \Vdash^{\forall} A\to C, b\Vdash^{\exists} A, b\Vdash^{\forall} A\to B, a\Vdash^{\exists} A, \cdots\Rightarrow x\Vdash_a A|B\wedge C}\ \mathsf{Tr}_1$$

$$\dfrac{}{x\Vdash_b A|C, b\in N(x), b\subseteq a, a\in N(x), b\Vdash^{\exists} A, b\Vdash^{\forall} A\to B, a\Vdash^{\exists} A, \cdots\Rightarrow x\Vdash_a A|B\wedge C}\ \mathsf{L}|$$

$$\dfrac{\triangledown \qquad x\Vdash_b A|C, b\in N(x), b\subseteq a, a\in N(x), b\Vdash^{\exists} A, b\Vdash^{\forall} A\to B, a\Vdash^{\exists} A, \cdots\Rightarrow x\Vdash_a A|B\wedge C}{b\in N(x), b\subseteq a, a\in N(x), b\Vdash^{\exists} A, b\Vdash^{\forall} A\to B, a\Vdash^{\exists} A, x:A>B, x:A>C\Rightarrow x\Vdash_a A|B\wedge C}\ \mathsf{L}>$$

$$\dfrac{}{x\Vdash_a A|B, a\in N(x), a\Vdash^{\exists} A, x:A>B, x:A>C\Rightarrow x\Vdash_a A|B\wedge C}\ \mathsf{L}|$$

$$\dfrac{a\Vdash^{\exists} A\cdots \overset{\triangledown}{\Rightarrow} \ldots a\Vdash^{\exists} A \qquad x\Vdash_a A|B, a\in N(x), a\Vdash^{\exists} A, x:A>B, x:A>C\Rightarrow x\Vdash_a A|B\wedge C}{a\in N(x), a\Vdash^{\exists} A, x:A>B, x:A>C\Rightarrow x\Vdash_a A|B\wedge C}\ \mathsf{L}>$$

$$\dfrac{\dfrac{\dfrac{a\in N(x), a\Vdash^{\exists} A, x:A>B, x:A>C\Rightarrow x\Vdash_a A|B\wedge C}{x:A>B, x:A>C\Rightarrow x:A>(B\wedge C)}\ \mathsf{R}>}{x:(A>B)\wedge(A>C)\Rightarrow x:A>(B\wedge C)}\ \mathsf{R}\wedge}{\Rightarrow x:(A>B)\wedge(A>C)\to(A>(B\wedge C))}\ \mathsf{R}\to$$

We have omitted writing the derivable left premisses of two occurrences of $\mathsf{L}>$, which are: $c\Vdash^{\exists} A\cdots\Rightarrow\ldots c\Vdash^{\exists} A$ (lower occurrence) and $c\Vdash^{\exists} A\cdots\Rightarrow\ldots c\Vdash^{\exists} A$ (upper occurrence).

(ID)

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\dfrac{y \in a \ldots y : \overset{\triangledown}{A} \Rightarrow y : A \ldots}{y \in a \ldots y : A \Rightarrow x \Vdash_a A|A, a \Vdash^{\exists} A}\;\mathsf{R}\Vdash^{\exists}
}{\ldots a \Vdash^{\exists} A \Rightarrow x \Vdash_a A|A, a \Vdash^{\exists} A}\;\mathsf{L}\Vdash^{\exists}
\qquad
\dfrac{
\dfrac{k \in a \ldots k : A \overset{\triangledown}{\Rightarrow} x \Vdash_a A|A, k : A}{k \in a \cdots \Rightarrow x \Vdash_a A|A, k : A \to A}\;\mathsf{R}\to
}{a \subseteq a, a \in N(x), a \Vdash^{\exists} A \Rightarrow x \Vdash_a A|A, a \Vdash^{\forall} A \to A}\;\mathsf{R}\Vdash^{\forall}
}{a \subseteq a, a \in N(x), a \Vdash^{\exists} A \Rightarrow x \Vdash_a A|A}\;\mathsf{R}|
}{
\dfrac{a \in N(x), a \Vdash^{\exists} A \Rightarrow x \Vdash_a A|A}{\Rightarrow x : A > A}\;\mathsf{R}>
}\;\mathsf{Ref}
$$

(CM)

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\dfrac{\ldots y : C \overset{\triangledown}{\Rightarrow} y : C \ldots \qquad \ldots y : A \wedge \overset{\triangledown}{B} \Rightarrow y : A \ldots}{y \in c, c \subseteq a, c \subseteq b, c \Vdash^{\exists} A, y : A \to C, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, a \Vdash^{\exists} A \wedge B, y : A \wedge B \cdots \Rightarrow \ldots y : C}\;\mathsf{L}\to
}{y \in c, c \subseteq a, c \subseteq b, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, y : A \wedge B \cdots \Rightarrow \ldots y : C}\;\mathsf{L}\Vdash^{\forall}
}{y \in c, c \subseteq a, c \subseteq b, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots y : A \wedge B \to C}\;\mathsf{R}\to
}{c \subseteq a \ldots c \subseteq b, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots c \Vdash^{\forall} A \wedge B \to C}\;\mathsf{R}\Vdash^{\forall}
}{c \subseteq a, c \subseteq b, c \Vdash^{\exists} A, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C}\;\mathsf{R}|
}{c \in N(x), c \subseteq b, b \in N(x), b \subseteq a, a \in N(x), c \Vdash^{\exists} A, c \Vdash^{\exists} A \to C, b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C}\;\mathsf{Tr}
}{x \Vdash_b A|C, b \in N(x), b \subseteq a, a \in N(x), b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C}\;\mathsf{L}|
}{b \in N(x), b \subseteq a, a \in N(x), b \Vdash^{\exists} A, b \Vdash^{\exists} A \to B, a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C}\;\mathsf{L}>
}{x \Vdash_a A|B, a \in N(x), a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C}\;\mathsf{L}|
}{
\dfrac{
\dfrac{
\dfrac{a \in N(x), a \Vdash^{\exists} A \wedge B, x : A > B, x : A > C \Rightarrow x \Vdash_a A \wedge B|C}{x : A > B, x : A > C \Rightarrow x : (A \wedge B) > C}\;\mathsf{R}>
}{x : (A > B) \wedge (A > C) \Rightarrow x : (A \wedge B) > C}\;\mathsf{L}\wedge
}{\Rightarrow x : (A > B) \wedge (A > C) \to ((A \wedge B) > C)}\;\mathsf{R}\to
}\;\mathsf{L}>
$$

(In the printed tree, the conclusion of $\mathsf{L}|$ has an additional premise $\triangledown$ under the upper $\mathsf{L}>$ and the lower $\mathsf{L}>$ likewise.)

We have omitted three derivable left premisses: $a \Vdash^{\exists} a \wedge B \cdots \Rightarrow \ldots a \Vdash^{\exists} A$, premiss of the lower occurrence of $\mathsf{L}>$, $b \Vdash^{\exists} A \cdots \Rightarrow \ldots b \Vdash^{\exists} A$ premiss of the upper occurrence of $\mathsf{L}>$, and $c \subseteq a, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots c \Vdash^{\exists} A \wedge B$, premiss of $\mathsf{R}|$.

(RT)

$$\dfrac{\dfrac{\overset{\triangledown}{\dots y : C \Rightarrow y : C \dots} \quad \dfrac{\overset{\triangledown}{\dots y : B, y : A \Rightarrow y : C, y : A \wedge B} \quad \overset{\triangledown}{\dots y : A \Rightarrow y : C, y : A \wedge B, y : A}}{\dots y : A \to B, y : A \Rightarrow y : C, y : A \wedge B}\ \mathsf{L}\to}{\dots y : A \wedge B \to C, y : A \to B, y : A \Rightarrow y : C}\ \mathsf{L}\to}{y \in b, y \in c, c \subseteq a, c \subseteq b, b \subseteq a \dots c \Vdash^{\exists} A \wedge B, y : A \wedge B \to C, b \Vdash^{\exists} A, y : A \to B, a \Vdash^{\exists} A \Rightarrow y : A \to C}\ \mathsf{R}\to$$

$$\dfrac{y \in b, y \in c, c \subseteq a, c \subseteq b, b \subseteq a \dots c \Vdash^{\exists} A \wedge B, y : A \wedge B \to C, b \Vdash^{\exists} A, y : A \to B, a \Vdash^{\exists} A \Rightarrow y : A \to C}{y \in b, y \in c, c \subseteq a, c \subseteq b, b \subseteq a \dots c \Vdash^{\exists} A \wedge B, y : A \wedge B \to C, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, a \Vdash^{\exists} A \Rightarrow y : A \to C}\ \mathsf{L}\Vdash^{\forall}$$

$$\dfrac{}{y \in c, c \subseteq a, c \subseteq b, b \subseteq a \dots c \Vdash^{\exists} A \wedge B, y : A \wedge B \to C, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, a \Vdash^{\exists} A \Rightarrow y : A \to C}\ \mathsf{L}\subseteq$$

$$\dfrac{}{y \in c, c \subseteq a, c \subseteq b, b \subseteq a \dots c \Vdash^{\exists} A \wedge B, c \Vdash^{\forall} A \wedge B \to C, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, a \Vdash^{\exists} A \Rightarrow y : A \to C}\ \mathsf{L}\Vdash^{\forall}$$

$$\dfrac{\overset{\triangledown}{\dots c \Vdash^{\exists} A \Rightarrow c \Vdash^{\exists} A \dots} \qquad \dfrac{}{c \subseteq a, c \subseteq b, b \subseteq a \dots c \Vdash^{\exists} A \wedge B, c \Vdash^{\forall} A \wedge B \to C, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, a \Vdash^{\exists} A \Rightarrow c \Vdash^{\forall} A \to C}\ \mathsf{R}\Vdash^{\forall}}{c \subseteq a, c \subseteq b, b \subseteq a \dots c \Vdash^{\exists} A \wedge B, c \Vdash^{\forall} A \wedge B \to C, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, a \Vdash^{\exists} A \dots \Rightarrow x \Vdash_a A|C}\ \mathsf{R}|$$

$$\dfrac{}{c \in N(x), c \subseteq b, b \in N(x), b \subseteq a, a \in N(x), c \Vdash^{\exists} A \wedge B, c \Vdash^{\forall} A \wedge B \to C, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, a \Vdash^{\exists} A \dots \Rightarrow x \Vdash_a A|C}\ \mathsf{Tr}_1$$

$$\dfrac{}{x \Vdash_b A \wedge B|C, b \in N(x), b \subseteq a, a \in N(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, a \Vdash^{\exists} A \dots \Rightarrow x \Vdash_a A|C}\ \mathsf{L}|$$

$$\dfrac{\triangledown \qquad x \Vdash_b A \wedge B|C, b \in N(x), b \subseteq a, a \in N(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, a \Vdash^{\exists} A \dots \Rightarrow x \Vdash_a A|C}{b \in N(x), b \subseteq a, a \in N(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, a \Vdash^{\exists} A, x : A > B, x : (A \wedge B) > C \Rightarrow x \Vdash_a A|C}\ \mathsf{L}>$$

$$\dfrac{}{x \Vdash_a A|B, a \in N(x), a \Vdash^{\exists} A, x : A > B, x : (A \wedge B) > C \Rightarrow x \Vdash_a A|C}\ \mathsf{L}|$$

$$\dfrac{\triangledown \qquad x \Vdash_a A|B, a \in N(x), a \Vdash^{\exists} A, x : A > B, x : (A \wedge B) > C \Rightarrow x \Vdash_a A|C}{a \in N(x), a \Vdash^{\exists} A, x : A > B, x : (A \wedge B) > C \Rightarrow x \Vdash_a A|C}\ \mathsf{L}>$$

$$\dfrac{\dfrac{\dfrac{a \in N(x), a \Vdash^{\exists} A, x : A > B, x : (A \wedge B) > C \Rightarrow x \Vdash_a A|C}{x : A > B, x : (A \wedge B) > C \Rightarrow x : A > C}\ \mathsf{R}>}{x : (A > B) \wedge ((A \wedge B) > C) \Rightarrow x : A > C}\ \mathsf{L}\wedge}{\Rightarrow x : (A > B) \wedge ((A \wedge B) > C) \to (A > C)}\ \mathsf{R}\to$$

We have omitted writing the derivable left premiss of the lowest occurrence of $\mathsf{L}>$, which is sequent $\dots a \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A \dots$. We have omitted the premiss of the second lowes occurrence of the same rule, which is sequent $\dots b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B \Rightarrow \dots b \Vdash^{\exists} A \wedge B$.

(OR) Derivation of (OR) is the longest of all axiom derivations. Derivation of sequent $(*)$ can be found in next page.

$$
\begin{prooftree}
\AxiomC{$\dots a \Vdash^{\exists} A \vee B \overset{\triangledown}{\Rightarrow} \dots a \Vdash^{\exists} A, a \Vdash^{\exists} B$}
\AxiomC{$\dots b \Vdash^{\exists} A \overset{\triangledown}{\Rightarrow} \dots a \Vdash^{\exists} A \vee B$}
\AxiomC{$\dots y : C \overset{\triangledown}{\Rightarrow} \dots y : C$}
\AxiomC{$\dots y : A \vee B \overset{\triangledown}{\Rightarrow} \dots y : A, y : B$}
\RightLabel{$\mathsf{L} \rightarrow$}
\BinaryInfC{$\dots y \in a, y \in b, b \subseteq a, b \Vdash^{\exists} B, y : B \rightarrow C, y : A \vee B \Rightarrow \dots y : A, y : C$}
\RightLabel{$\mathsf{L} \subseteq, \mathsf{R} \Vdash^{\exists}$}
\UnaryInfC{$\dots y \in b, b \subseteq a, b \Vdash^{\exists} B, y : B \rightarrow C, y : A \vee B \Rightarrow \dots a \Vdash^{\exists} A, y : C$}
\RightLabel{$\mathsf{L} \Vdash^{\forall}$}
\UnaryInfC{$\dots y \in b, b \subseteq a, b \Vdash^{\exists} B, b \Vdash^{\forall} B \rightarrow C, y : A \vee B \Rightarrow \dots a \Vdash^{\exists} A, y : C$}
\RightLabel{$\mathsf{R} \rightarrow$}
\UnaryInfC{$\dots y \in b, b \subseteq a, b \Vdash^{\exists} B, b \Vdash^{\forall} B \rightarrow C \Rightarrow \dots a \Vdash^{\exists} A, y : (A \vee B) \rightarrow C$}
\RightLabel{$\mathsf{R} \Vdash^{\forall}$}
\UnaryInfC{$\dots b \subseteq a, b \Vdash^{\exists} B, b \Vdash^{\forall} B \rightarrow C \Rightarrow \dots a \Vdash^{\exists} A, b \Vdash^{\forall} (A \vee B) \rightarrow C$}
\RightLabel{$\mathsf{R}|$}
\BinaryInfC{$\dots b \subseteq a, b \Vdash^{\exists} B, b \Vdash^{\forall} B \rightarrow C \Rightarrow x \Vdash_{a} A \vee B | C, a \Vdash^{\exists} A$}
\RightLabel{$\mathsf{L}|$}
\UnaryInfC{$\dots x \Vdash_{a} B | C \Rightarrow x \Vdash_{a} A \vee B | C, a \Vdash^{\exists} A$}
\RightLabel{$\mathsf{L} >$}
\BinaryInfC{$a \in N(x), a \Vdash^{\exists} A \vee B, x : A > C, x : B > C \Rightarrow x \Vdash_{a} A \vee B | C, a \Vdash^{\exists} A$}
\AxiomC{$(*)$}
\RightLabel{$\mathsf{L} >$}
\BinaryInfC{$a \in N(x), a \Vdash^{\exists} A \vee B, x : A > C, x : B > C \Rightarrow x \Vdash_{a} A \vee B | C$}
\RightLabel{$\mathsf{R} >$}
\UnaryInfC{$x : A > C, x : B > C \Rightarrow x : (A \vee B) > C$}
\RightLabel{$\mathsf{L} \wedge$}
\UnaryInfC{$x : (A > C) \wedge (B > C) \Rightarrow x : (A \vee B) > C$}
\RightLabel{$\mathsf{R} \rightarrow$}
\UnaryInfC{$\Rightarrow x : (A > C) \wedge (B > C) \rightarrow (A \vee B) > C$}
\end{prooftree}
$$

Here follows the second part of the derivation of (OR): derivation of $(*)$, in which we omitted writing the derivation of sequent $(**)$, the left premiss of $\mathsf{R}|$, which corresponds to the following sequent:
$(**)$ $c \subseteq a, c \in N(x), c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, x : B > Cx : A > C, X : B > C \Rightarrow x \Vdash_a A \vee B|C, c \Vdash^{\forall} (A \vee B) \to C, c \Vdash^{\exists} B.$

$$
\dfrac{\dfrac{\ldots c \Vdash^{\exists} A \overset{\triangledown}{\Rightarrow} \ldots c \Vdash^{\exists} A \vee B \qquad \dfrac{(**) \qquad \dfrac{\dfrac{\dfrac{\ldots d \Vdash^{\exists} A \overset{\triangledown}{\Rightarrow} d \Vdash^{\exists} A \ldots \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{k : C \overset{\triangledown}{\Rightarrow} k : C \qquad \dfrac{k : C \overset{\triangledown}{\Rightarrow} k : C \quad k : A \vee B \overset{\triangledown}{\Rightarrow} k : B, k : A}{\ldots k : A \to C, k : A \vee B \Rightarrow \ldots k : C, k : B}\ \mathsf{L}\to}{\ldots k : B \to C, k : A \to C, k : A \vee B \Rightarrow \ldots k : C}\ \mathsf{L}\to}{\ldots k : B \to C, k : A \to C \Rightarrow \ldots k : (A \vee B) \to C}\ \mathsf{R}\to}{\ldots k \in c, k \in d, d \subseteq c, d \Vdash^{\forall} B \to C, c \subseteq a, c \Vdash^{\forall} A \to C \Rightarrow \ldots k : (A \vee B) \to C}\ \mathsf{L}\Vdash^{\forall}(2\mathsf{x})}{\ldots k \in d, d \subseteq c, d \Vdash^{\forall} B \to C, c \Vdash^{\forall} A \to C \Rightarrow \ldots k : (A \vee B) \to C}\ \mathsf{L}\subseteq}{\ldots d \subseteq c, d \Vdash^{\forall} B \to C, c \Vdash^{\forall} A \to C \Rightarrow \ldots d \Vdash^{\forall} (A \vee B) \to C}\ \mathsf{R}\Vdash^{\forall}}{\ldots d \subseteq a, d \subseteq c, d \Vdash^{\exists} B, d \Vdash^{\forall} B \to C, c \subseteq a, c \Vdash^{\forall} A \to C \Rightarrow \ldots x \Vdash_a A \vee B|C}\ \mathsf{R}|}{\ldots d \subseteq c, d \Vdash^{\exists} B, d \Vdash^{\forall} B \to C, c \subseteq a, c \Vdash^{\forall} A \to C \Rightarrow \ldots x \Vdash_a A \vee B|C}\ \mathsf{L}\subseteq}{\ldots x \Vdash_c B|C, c \subseteq a, c \Vdash^{\forall} A \to C \Rightarrow \ldots x \Vdash_a A \vee B|C}\ \mathsf{L}|}{c \subseteq a, c \in N(x), c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, x : B > C \cdots \Rightarrow x \Vdash_a A \vee B|C, c \Vdash^{\forall} (A \vee B) \to C}\ \mathsf{L}>}{\ldots c \subseteq a, c \in N(x), c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, x : B > C \Rightarrow x \Vdash_a A \vee B|C}\ \mathsf{R}|}{x \Vdash_a A|C, a \in N(x), a \Vdash^{\exists} A \vee B, x : A > C, x : B > C \Rightarrow x \Vdash_a A \vee B|C}\ \mathsf{L}|
$$

(N)

$$
\dfrac{\dfrac{\dfrac{\dfrac{y \in a, a \in N(x), x : \top > \bot \Rightarrow a \Vdash^{\exists} \top, y : \top}{y \in a, a \in N(x), x : \top > \bot \Rightarrow a \Vdash^{\exists} \top}\ \mathsf{R}\Vdash^{\exists}}{a \in N(x), x : \top > \bot \Rightarrow a \Vdash^{\exists} \top}\ \mathsf{0} \qquad \dfrac{\dfrac{\dfrac{\cdots \Rightarrow y : \top \qquad y : \bot \Rightarrow \ldots}{c \in N(x), c \subseteq a, a \Vdash^{\exists} \top, a \Vdash^{\forall} \top \to \bot, y : \top \to \bot, y \in a, a \in N(x), x : \top > \bot \Rightarrow x : \bot}\ \mathsf{L}\subseteq}{c \in N(x), c \subseteq a, a \Vdash^{\exists} \top, a \Vdash^{\forall} \top \to \bot, y \in a, a \in N(x), x : \top > \bot \Rightarrow x : \bot}\ \mathsf{L}\Vdash^{\forall}}{x \Vdash_a \top | \bot, y \in a, a \in N(x), x : \top > \bot \Rightarrow x : \bot}\ \mathsf{L}|}{\dfrac{a \in N(x), x : \top > \bot \Rightarrow}{\dfrac{x : \top > \bot \Rightarrow}{\Rightarrow x : \neg(\top > \bot)}\ \mathsf{R}\neg}\ \mathsf{N}}\ \mathsf{L}>
$$

(T)

$$
\dfrac{\dfrac{\dfrac{x \in a, a \in N(x), x : A, x : \overset{\triangledown}{A} > \bot \Rightarrow x : \bot, a \Vdash^{\exists} A, x : A}{x \in a, a \in N(x), x : A, x : A > \bot \Rightarrow x : \bot, a \Vdash^{\exists} A}\ \mathsf{R}\Vdash^{\exists} \qquad \dfrac{\dfrac{\dfrac{\dfrac{y : A \to \bot, y \in b, y \overset{\triangledown}{:} A \cdots \Rightarrow x : \bot, y : A \qquad y : \bot \cdots \Rightarrow}{y : A \to \bot, y \in b, y : A, b \in N(x), b \subseteq a, x \in a, a \in N(x), x : A > \bot \Rightarrow x : \bot}\ \mathsf{L}\to}{y \in b, y : A, b \in N(x), b \subseteq a, b \Vdash^{\forall} A \to \bot, x \in a, a \in N(x), x : A > \bot \Rightarrow x : \bot}\ \mathsf{L}\Vdash^{\forall}}{b \in N(x), b \subseteq a, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to \bot, x \in a, a \in N(x), x : A > \bot \Rightarrow x : \bot}\ \mathsf{L}\Vdash^{\exists}}{x \Vdash_a A | \bot, x \in a, a \in N(x), x : A, x : A > \bot, \Rightarrow x : \bot}\ \mathsf{L}\subseteq}{\dfrac{x \in a, a \in N(x), x : A, x : A > \bot \Rightarrow x : \bot}{\dfrac{x : A, x : A < \bot \Rightarrow x : \bot}{\dfrac{x : A \Rightarrow x : (A > \bot) \to \bot}{\Rightarrow x : A \to ((A > \bot) \to \bot)}\ \mathsf{R}\to}\ \mathsf{R}\to}\ \mathsf{T}}\ \mathsf{L}>
$$

(W)

$$\dfrac{x \in b,\ \dots x : A \overset{\triangledown}{\Rightarrow} x : B, x : A \qquad x : B \cdots \overset{\triangledown}{\Rightarrow} x : B}{x : A \to B, x \in b, z \in b, b \in N(x), b \subseteq a, z : A, b \Vdash^{\forall} A \to B, x \in a, a \in N(x), x : A > B, x : A \Rightarrow x : B}\ \mathsf{L}\to$$

$$\dfrac{x : A \to B, x \in b, z \in b, b \in N(x), b \subseteq a, z : A, b \Vdash^{\forall} A \to B, x \in a, a \in N(x), x : A > B, x : A \Rightarrow x : B}{x \in b, z \in b, b \in N(x), b \subseteq a, z : A, b \Vdash^{\forall} A \to B, x \in a, a \in N(x), x : A > B, x : A \Rightarrow x : B}\ \mathsf{L}\Vdash^{\forall}$$

$$\dfrac{x \in b, z \in b, b \in N(x), b \subseteq a, z : A, b \Vdash^{\forall} A \to B, x \in a, a \in N(x), x : A > B, x : A \Rightarrow x : B}{z \in b, b \in N(x), b \subseteq a, z : A, b \Vdash^{\forall} A \to B, x \in a, a \in N(x), x : A > B, x : A \Rightarrow x : B}\ \mathsf{W}$$

$$\dfrac{z \in b, b \in N(x), b \subseteq a, z : A, b \Vdash^{\forall} A \to B, x \in a, a \in N(x), x : A > B, x : A \Rightarrow x : B}{b \in N(x), b \subseteq a, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, x \in a, a \in N(x), x : A > B, x : A \Rightarrow x : B}\ \mathsf{L}\Vdash^{\exists}$$

$$\dfrac{b \in N(x), b \subseteq a, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, x \in a, a \in N(x), x : A > B, x : A \Rightarrow x : B}{x \Vdash_a A|B, x \in a, a \in N(x), x : A > B, x : A \Rightarrow x : B}\ \mathsf{L}|$$

$$\dfrac{\dfrac{x : A \cdots \overset{\triangledown}{\Rightarrow} \dots x : A}{x \in a, x : A, \cdots \Rightarrow x : B, a \Vdash^{\exists} A}\ \mathsf{R}\Vdash^{\exists} \qquad x \Vdash_a A|B, x \in a, a \in N(x), x : A > B, x : A \Rightarrow x : B}{x \in a, a \in N(x), x : A > B, x : A \Rightarrow x : B}\ \mathsf{L}>$$

$$\dfrac{\dfrac{\dfrac{\dfrac{x \in a, a \in N(x), x : A > B, x : A \Rightarrow x : B}{a \in N(x), x : A > B, x : A \Rightarrow x : B}\ \mathsf{W}}{x : A > B, x : A \Rightarrow x : B}\ {}_{N}}{x : A > B \Rightarrow A \to B}\ \mathsf{R}\to}{\Rightarrow x : (A > B) \to (A \to B)}\ \mathsf{R}\to$$

(C)

$$\dfrac{\dfrac{x : A \cdots \overset{\triangledown}{\Rightarrow} \dots x : A}{x \in \{x\}, \{x\} \in N(x), \{x\} \subseteq a, a \Vdash^{\exists} A, x : A, x : B \Rightarrow \dots \{x\} \Vdash^{\exists} A}\ \mathsf{R}\Vdash^{\exists}}{\{x\} \in N(x), \{x\} \subseteq a, a \Vdash^{\exists} A, x : A, x : B \Rightarrow \dots \{x\} \Vdash^{\exists} A}\ \mathsf{Single}$$

$$\dfrac{y : B \cdots \overset{\triangledown}{\Rightarrow} \dots y : B}{y : A, y \in \{x\}, \{x\} \in N(x), a \in N(x), \{x\} \subseteq a, a \Vdash^{\exists} A, x : A, x : B \Rightarrow x \Vdash_a A|B, y : B}\ \mathsf{Repl}_1$$

$$\dfrac{y : A, y \in \{x\}, \{x\} \in N(x), a \in N(x), \{x\} \subseteq a, a \Vdash^{\exists} A, x : A, x : B \Rightarrow x \Vdash_a A|B, y : B}{y \in \{x\}, \{x\} \in N(x), a \in N(x), \{x\} \subseteq a, a \Vdash^{\exists} A, x : A, x : B \Rightarrow x \Vdash_a A|B, y : A \to B}\ \mathsf{R}\to$$

$$\dfrac{y \in \{x\}, \{x\} \in N(x), a \in N(x), \{x\} \subseteq a, a \Vdash^{\exists} A, x : A, x : B \Rightarrow x \Vdash_a A|B, y : A \to B}{\{x\} \in N(x), a \in N(x), \{x\} \subseteq a, a \Vdash^{\exists} A, x : A, x : B \Rightarrow x \Vdash_a A|B, \{x\} \Vdash^{\forall} A \to B}\ \mathsf{R}\Vdash^{\forall}$$

$$\dfrac{\{x\} \in N(x), \{x\} \subseteq a, a \Vdash^{\exists} A, x : A, x : B \Rightarrow \dots \{x\} \Vdash^{\exists} A \qquad \{x\} \in N(x), a \in N(x), \{x\} \subseteq a, a \Vdash^{\exists} A, x : A, x : B \Rightarrow x \Vdash_a A|B, \{x\} \Vdash^{\forall} A \to B}{\{x\} \in N(x), \{x\} \subseteq a, a \in N(x), a \Vdash^{\exists} A, x : A, x : B \Rightarrow x \Vdash_a A|B}\ \mathsf{R}|$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\{x\} \in N(x), \{x\} \subseteq a, a \in N(x), a \Vdash^{\exists} A, x : A, x : B \Rightarrow x \Vdash_a A|B}{a \in N(x), a \Vdash^{\exists} A, x : A, x : B \Rightarrow x \Vdash_a A|B}\ \mathsf{C}}{x : A, x : B \Rightarrow x : A > B}\ \mathsf{R}>}{x : A \wedge B \Rightarrow x : A > B}\ \mathsf{L}\wedge}{\Rightarrow x : (A \wedge B) \to (A > B)}\ \mathsf{R}\to$$

(U$_1$)

$$\dfrac{\dfrac{\overset{\nabla}{\ldots k : \neg A \Rightarrow \ldots k : \neg A} \qquad \overset{\mathsf{init}}{\ldots k : \bot \Rightarrow \ldots}}{k \in d, d \subseteq c \ldots k : \neg A, d \Vdash^{\forall} \neg A \to \bot, k : \neg A \to \bot, z : \neg A, x : \neg A > \bot \Rightarrow \ldots}\ \mathsf{L}\to}{k \in d, d \subseteq c \ldots k : \neg A, d \Vdash^{\forall} \neg A \to \bot, z : \neg A, x : \neg A > \bot \Rightarrow \ldots}\ \mathsf{L}\Vdash^{\forall}$$

$$\dfrac{\dfrac{k \in d, d \subseteq c \ldots k : \neg A, d \Vdash^{\forall} \neg A \to \bot, z : \neg A, x : \neg A > \bot \Rightarrow \ldots}{d \subseteq c, d \in N(x) \ldots d \Vdash^{\exists} \neg A, d \Vdash^{\forall} \neg A \to \bot, z : \neg A, x : \neg A > \bot \Rightarrow \ldots}\ \mathsf{L}\Vdash^{\exists}}{x \Vdash_c \neg A | \bot \ldots z : \neg A, x : \neg A > \bot \Rightarrow \ldots}\ \mathsf{L}|$$

$$\dfrac{\overset{\nabla}{z \in c, z \in b \ldots z : \neg A \Rightarrow \ldots c \Vdash^{\exists} \neg A, z : \neg A}}{z \in c, z \in b \ldots z : \neg A \Rightarrow \ldots c \Vdash^{\exists} \neg A}\ \mathsf{R}\Vdash^{\exists}$$

$$\dfrac{z \in c, z \in b \ldots z : \neg A \Rightarrow \ldots c \Vdash^{\exists} \neg A \qquad x \Vdash_c \neg A | \bot \ldots z : \neg A, x : \neg A > \bot \Rightarrow \ldots}{z \in c, z \in b, b \in N(y), y \in a, a \in N(x), z : \neg A, x : \neg A > \bot \Rightarrow x \Vdash_a \neg(\neg A > \bot) > \bot, y \Vdash_b \neg A | \bot}\ \mathsf{L}>$$

$$\dfrac{z \in c, z \in b, b \in N(y), y \in a, a \in N(x), z : \neg A, x : \neg A > \bot \Rightarrow x \Vdash_a \neg(\neg A > \bot) > \bot, y \Vdash_b \neg A | \bot}{z \in b, b \in N(y), y \in a, a \in N(x), z : \neg A, x : \neg A > \bot \Rightarrow x \Vdash_a \neg(\neg A > \bot) > \bot, y \Vdash_b \neg A | \bot}\ \mathsf{U}_1$$

$$\dfrac{z \in b, b \in N(y), y \in a, a \in N(x), z : \neg A, x : \neg A > \bot \Rightarrow x \Vdash_a \neg(\neg A > \bot) > \bot, y \Vdash_b \neg A | \bot}{b \in N(y), y \in a, a \in N(x), b \Vdash^{\exists} \neg A, x : \neg A > \bot \Rightarrow x \Vdash_a \neg(\neg A > \bot) > \bot, y \Vdash_b \neg A | \bot}\ \mathsf{L}\Vdash^{\exists}$$

$$\dfrac{b \in N(y), y \in a, a \in N(x), b \Vdash^{\exists} \neg A, x : \neg A > \bot \Rightarrow x \Vdash_a \neg(\neg A > \bot) > \bot, y \Vdash_b \neg A | \bot}{y \in a, a \in N(x), x : \neg A > \bot \Rightarrow x \Vdash_a \neg(\neg A > \bot) > \bot, y : \neg A > \bot}\ \mathsf{R}>$$

$$\dfrac{y \in a, a \in N(x), x : \neg A > \bot \Rightarrow x \Vdash_a \neg(\neg A > \bot) > \bot, y : \neg A > \bot}{y \in a, a \in N(x), y : \neg(\neg A > \bot), x : \neg A > \bot \Rightarrow x \Vdash_a \neg(\neg A > \bot) > \bot}\ \mathsf{L}\neg$$

$$\dfrac{y \in a, a \in N(x), y : \neg(\neg A > \bot), x : \neg A > \bot \Rightarrow x \Vdash_a \neg(\neg A > \bot) > \bot}{a \in N(x), a \Vdash^{\exists} \neg(\neg A > \bot), x : \neg A > \bot \Rightarrow x \Vdash_a \neg(\neg A > \bot) > \bot}\ \mathsf{L}\Vdash^{\exists}$$

$$\dfrac{\dfrac{a \in N(x), a \Vdash^{\exists} \neg(\neg A > \bot), x : \neg A > \bot \Rightarrow x \Vdash_a \neg(\neg A > \bot) > \bot}{x : \neg A > \bot \Rightarrow x : \neg(\neg A > \bot) > \bot}\ \mathsf{R}>}{\Rightarrow x : (\neg A > \bot) \to (\neg(\neg A > \bot) > \bot)}\ \mathsf{R}\to$$

(U_2)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\nabla}{y \in c, y \in a, y : A \cdots \Rightarrow \ldots c \Vdash^{\exists} A, y : A}}{y \in c, y : A \cdots \Rightarrow \ldots c \Vdash^{\exists} A}\ \mathsf{R} \Vdash^{\exists} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\nabla}{k : A \cdots \Rightarrow \ldots k : A} \qquad \dfrac{\nabla}{k : \bot \cdots \Rightarrow \ldots}}{k \in d, d \subseteq c, d \in N(z) \ldots k : A, k : A \subseteq \bot, y : A, z : A > \bot \Rightarrow \ldots}\ \mathsf{L} \rightarrow}{k \in d, d \subseteq c, d \in N(z) \ldots k : A, z \Vdash^{\forall} A \subseteq \bot, y : A, z : A > \bot \Rightarrow \ldots}\ \mathsf{L} \Vdash^{\forall}}{d \subseteq c, d \in N(z) \ldots z \Vdash^{\exists} A, z \Vdash^{\forall} A \subseteq \bot, y : A, z : A > \bot \Rightarrow \ldots}\ \mathsf{L} \Vdash^{\exists}}{z \Vdash_c A|\bot \ldots y : A, z : A > \bot \Rightarrow \ldots}\ \mathsf{L}|}{y \in c, c \in N(z), z \in b, b \in N(x), y \in a, a \in N(x), y : A, z : A > \bot \Rightarrow x \Vdash_a A|\bot, x \Vdash_b A > \bot|\bot}\ \mathsf{L} >}{z \in b, b \in N(x), y \in a, a \in N(x), y : A, z : A > \bot \Rightarrow x \Vdash_a A|\bot, x \Vdash_b A > \bot|\bot}\ \mathsf{U}_2}{b \in N(x), y \in a, a \in N(x), y : A, b \Vdash^{\exists} A > \bot \Rightarrow x \Vdash_a A|\bot, x \Vdash_b A > \bot|\bot}\ \mathsf{L} \Vdash^{\exists}}{y \in a, a \in N(x), y : A \Rightarrow x \Vdash_a A|\bot, x : (A > \bot) > \bot}\ \mathsf{R} >}{a \in N(x), a \Vdash^{\exists} A \Rightarrow x \Vdash_a A|\bot, x : (A > \bot) > \bot}\ \mathsf{L} \Vdash^{\exists}}{\Rightarrow x : A > \bot, x : (A > \bot) > \bot}\ \mathsf{R} >}{x : \neg A > \bot \Rightarrow (A > \bot) > \bot}\ \mathsf{L}\neg}{x : \neg(A > \bot) \rightarrow ((A > \bot) > \bot)}\ \mathsf{R} \rightarrow
$$

(A$_1$)

$$
\dfrac{
a \Vdash^{\exists} C \cdots \overset{\triangledown}{\Rightarrow} \dots a \Vdash^{\exists} C
\qquad
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\dfrac{
\dfrac{
\dfrac{
b \Vdash^{\exists} A \cdots \overset{\triangledown}{\Rightarrow} \dots b \Vdash^{\exists} A
\qquad
\dfrac{
\dfrac{
\dfrac{
c \Vdash^{\exists} A \cdots \overset{\triangledown}{\Rightarrow} \dots c \Vdash^{\exists} A
\qquad
c \Vdash^{\forall} A \to B \cdots \overset{\triangledown}{\Rightarrow} c \Vdash^{\forall} A \to B
}{
c \in N(y), c \in N(x), c \subseteq b, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B, b \in N(x), b \in N(y) ... a \Vdash^{\exists} C, x : A > B \Rightarrow \dots
}\ \mathsf{R|}
}{
c \in N(x), c \subseteq b, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B, b \in N(x), b \in N(y), b \Vdash^{\exists} A ... a \Vdash^{\exists} C, x : A > B \Rightarrow \dots
}\ \mathsf{A_1}
}{
x \Vdash_b A|B, b \in N(x), b \in N(y), b \Vdash^{\exists} A, y \in a, a \subseteq a, a \in N(x), y : C, a \Vdash^{\exists} C, x : A > B \Rightarrow \dots
}\ {\scriptstyle LC}
}{
b \in N(x), b \in N(y), d \Vdash^{\exists} A, y \in a, a \subseteq a, a \in N(x), y : C, a \Vdash^{\exists} C, x : A > B \Rightarrow x \Vdash_a C|(A > B), y \Vdash_b A|B
}\ \mathsf{L>}
}{
b \in N(y), b \Vdash^{\exists} A, y \in a, a \subseteq a, a \in N(x), y : C, a \Vdash^{\exists} C, x : A > B \Rightarrow x \Vdash_a C|(A > B), y \Vdash_b A|B
}\ \mathsf{A_2}
}{
y \in a, a \subseteq a, a \in N(x), y : C, a \Vdash^{\exists} C, x : A > B \Rightarrow x \Vdash_a C|(A > B), y : A > B
}\ \mathsf{R>}
}{
y \in a, a \subseteq a, a \in N(x), a \Vdash^{\exists} C, x : A > B \Rightarrow x \Vdash_a C|(A > B), y : C \to A > B
}\ \mathsf{R\to}
}{
\Sigma \ \equiv \ a \subseteq a, a \in N(x), a \Vdash^{\exists} C, x : A > B \Rightarrow x \Vdash_a C|A > B, a \Vdash^{\forall} C \to A > B
}\ \mathsf{R\Vdash^{\forall}}
}{
\dfrac{
\dfrac{
\dfrac{
a \subseteq a, a \in N(x), a \Vdash^{\exists} C, x : A > B \Rightarrow x \Vdash_a C|A > B
}{
a \in N(x), a \Vdash^{\exists} C, x : A > B \Rightarrow x \Vdash_a A|B
}\ \mathsf{Ref}
}{
x : A > B \Rightarrow x : C > (A > B)
}\ \mathsf{R>}
}{
\Rightarrow x : A > B \to C > (A > B)
}\ \mathsf{R\to}
}\ \mathsf{R|}
$$

(A$_2$)

$$
\dfrac{
\dfrac{
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\dfrac{
\dfrac{
b \Vdash^{\exists} A \overset{\triangledown}{\Rightarrow} b \Vdash^{\exists} A
\qquad
\dfrac{
\dfrac{
\dfrac{
c \Vdash^{\exists} A \overset{\triangledown}{\Rightarrow} c \Vdash^{\exists} A
\qquad
c \Vdash^{\forall} A \to B \overset{\triangledown}{\Rightarrow} c \Vdash^{\exists} A \to B
}{
c \in N(x), c \in N(y), b \in N(y), b \in N(x), a \in N(x), a \subseteq a, y \in a, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B \cdots \Rightarrow \ldots x \Vdash_b A|B
}\ \mathsf{R|}
}{
c \in N(y), c \subseteq b, b \in N(y), b \in N(x), a \in N(x), a \subseteq a, y \in a, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B \cdots \Rightarrow \ldots x \Vdash_b A|B
}\ \mathsf{A_2}
}{
y \Vdash_b A|B, b \in N(y), b \in N(x), a \in N(x), a \subseteq a, y \in a, b \Vdash^{\exists} A, a \Vdash^{\exists} C, y : C, y : A > B \Rightarrow \ldots x \Vdash_b A|B
}\ \mathsf{L|}
}{
b \in N(y), b \in N(x), a \in N(x), a \subseteq a, y \in a, b \Vdash^{\exists} A, a \Vdash^{\exists} C, y : C, y : A > B \Rightarrow x \Vdash_a C|\neg(A > B), x \Vdash_b A|B
}\ \mathsf{L>}
}{
b \in N(x), a \in N(x), a \subseteq a, y \in a, b \Vdash^{\exists} A, a \Vdash^{\exists} C, y : C, y : A > B \Rightarrow x \Vdash_a C|\neg(A > B), x \Vdash_b A|B
}\ \mathsf{A_1}
}{
a \in N(x), a \subseteq a, y \in a, a \Vdash^{\exists} C, y : C, y : A > B \Rightarrow x \Vdash_a C|\neg(A > B), x : A > B
}\ \mathsf{R>}
}{
a \in N(x), a \subseteq a, y \in a, a \Vdash^{\exists} C, y : C \Rightarrow x \Vdash_a C|\neg(A > B), y : \neg(A > B), x : A > B
}\ \mathsf{R\neg}
}{
a \in N(x), a \subseteq a, y \in a, a \Vdash^{\exists} C \Rightarrow x \Vdash_a C|\neg(A > B), y : C \to \neg(A > B), x : A > B
}\ \mathsf{R\to}
}{
a \in N(x), a \subseteq a, a \Vdash^{\exists} C \Rightarrow x \Vdash_a C|\neg(A > B), a \Vdash^{\forall} C \to \neg(A > B), x : A > B
}\ \mathsf{R\Vdash^{\forall}}
\qquad
a \Vdash^{\exists} C \overset{\triangledown}{\Rightarrow} a \Vdash^{\exists} C
}{
a \in N(x), a \subseteq a, a \Vdash^{\exists} C \Rightarrow x \Vdash_a C|\neg(A > B), x : A > B
}\ \mathsf{R|}
}{
a \in N(x), a \Vdash^{\exists} C \Rightarrow x \Vdash_a C|\neg(A > B), x : A > B
}\ \mathsf{Ref}
}{
\Rightarrow x : C > \neg(A > B), x : A > B
}\ \mathsf{R>}
}{
x : \neg(A > B) \Rightarrow x : C > \neg(A > B)
}\ \mathsf{L\neg}
}{
\Rightarrow x : \neg(A > B) \to C > \neg(A > B)
}\ \mathsf{R\to}
$$

(CV) The upper derivation is the derivation of $(*)$, the rightmost premiss of Nes.

$$
\dfrac{\dfrac{\dots b \Vdash^{\exists} A \wedge B \overset{\triangledown}{\Rightarrow} \dots b \Vdash^{\exists} A \wedge B \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dots k : C \overset{\triangledown}{\Rightarrow} \dots k : C \quad \dots k : A \wedge B \overset{\triangledown}{\Rightarrow} \dots k : A}{\dots k \in c, k \in b, b \subseteq c, k : A \to C, k : A \wedge B :\Rightarrow \dots k : C}\,\mathsf{L}\to}{\dots k \in c, k \in b, b \subseteq c, k : A \to C \Rightarrow \dots k : (A \wedge B) \to C}\,\mathsf{R}\to}{\dots k \in b, b \subseteq c, c \Vdash^{\forall} A \to C \Rightarrow \dots k : (A \wedge B) \to C}\,\mathsf{L}\subseteq, \mathsf{L}\Vdash^{\forall}}{\dots b \subseteq c, c \Vdash^{\forall} A \to C \Rightarrow \dots b \Vdash^{\forall} (A \wedge B) \to C}\,\mathsf{R}\Vdash^{\forall}}{b \subseteq b, b \subseteq c, c \subseteq a, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, a \Vdash^{\exists} A, b \Vdash^{\exists} A \wedge B \Rightarrow x \Vdash_{b} A \wedge B|C, c \Vdash^{\forall} A \to \neg B}\,\mathsf{R}|}{(*)\ b \subseteq c, c \subseteq a, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, a \Vdash^{\exists} A, b \Vdash^{\exists} A \wedge B \Rightarrow x \Vdash_{b} A \wedge B|C, c \Vdash^{\forall} A \to \neg B}\,\mathsf{Ref}
$$

$$
\dfrac{\dfrac{\dfrac{\dots y \in c, y : A, y : B \overset{\triangledown}{\Rightarrow} \dots y : A \wedge B}{\dots y \in c \Rightarrow \dots y : A \to \neg B, y : A \wedge B}\,\mathsf{R}\to,\mathsf{R}\neg}{\dots y \in c \Rightarrow \dots y : A \to \neg B, c \Vdash^{\exists} A \wedge B}\,\mathsf{L}\Vdash^{\exists}}{\dots \Rightarrow \dots c \Vdash^{\forall} A \to \neg B, c \Vdash^{\exists} A \wedge B}\,\mathsf{R}\Vdash^{\exists}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{z : C \overset{\triangledown}{\Rightarrow} z : C \quad z : A \wedge B \overset{\triangledown}{\Rightarrow} z : A}{\dots z \in c, z : A \to C, z : A \wedge B \Rightarrow \dots z : C}\,\mathsf{L}\to}{\dots z \in c, c \Vdash^{\forall} A \to C, z : A \wedge B \Rightarrow \dots z : C}\,\Vdash^{\forall}}{\dots z \in c, c \Vdash^{\forall} A \to C \Rightarrow \dots z : (A \wedge B) \to C}\,\mathsf{R}\to}{\dots c \Vdash^{\forall} A \to C \Rightarrow \dots c \Vdash^{\forall} (A \wedge B) \to C}\,\mathsf{R}\Vdash^{\forall}
$$

$$
\dfrac{\dfrac{\text{(the two derivations above)}}{\dots c \subseteq b, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B, x : A > C \Rightarrow \dots x \Vdash_{b} A \wedge B|C, c \Vdash^{\forall} A \to \neg B}\,\mathsf{R}| \qquad (*)}{\dots c \subseteq a, c \in N(x), c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, x : A > C \Rightarrow \dots x \Vdash_{b} A \wedge B|C, c \Vdash^{\forall} A \to \neg B}\,\mathsf{Nes}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dots a \Vdash^{\exists} A \overset{\triangledown}{\Rightarrow} \dots a \Vdash^{\exists} A \qquad \dfrac{\dfrac{\dots c \Vdash^{\exists} A \overset{\triangledown}{\Rightarrow} \dots c \Vdash^{\exists} A \qquad \dots c \subseteq a, c \in N(x), c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, x : A > C \Rightarrow \dots x \Vdash_{b} A \wedge B|C, c \Vdash^{\forall} A \to \neg B}{\dots c \subseteq a, c \in N(x), c \Vdash^{\exists} A, c \Vdash^{\forall} A \to C, x : A > C \Rightarrow x \Vdash_{a} A|\neg B, x \Vdash_{b} A \wedge B|C}\,\mathsf{R}|}{\dots x \Vdash_{a} A > C, a \Vdash^{\exists} A, b \Vdash^{\exists} A \wedge B, x : A > C \Rightarrow x \Vdash_{a} A|\neg B, x \Vdash_{b} A \wedge B|C}\,\mathsf{L}|}{a \in N(x), b \in N(x), a \Vdash^{\exists} A, b \Vdash^{\exists} A \wedge B, x : A > C \Rightarrow x \Vdash_{a} A|\neg B, x \Vdash_{b} A \wedge B|C}\,\mathsf{L}>}{a \in N(x), a \Vdash^{\exists} A, x : A > C \Rightarrow x \Vdash_{a} A|\neg B, x : (A \wedge B) > C}\,\mathsf{R}>}{x : A > C \Rightarrow x : A > \neg B, x : (A \wedge B) > C}\,\mathsf{R}>}{\dfrac{\dfrac{x : A > C, x : \neg(A > \neg B) \Rightarrow x : (A \wedge B) > C}{x : (A > C) \wedge \neg(A > \neg B) \Rightarrow x : (A \wedge B) > C}\,\mathsf{L}\wedge}{\Rightarrow x : (A > C) \wedge \neg(A > \neg B) \to (A \wedge B) > C}\,\mathsf{R}\to}\,\mathsf{L}\neg
$$

Proof of Theorem 3.7.3.
To prove completeness of $\mathbf{G3V}^*$ with respect to the axiomatization for $\mathbb{V}$ and its extensions, we show that the inference rules are admissible in $\mathbf{G3V}^*$, and the axioms are derivable in the corresponding proof systems. We report the axiomatization, in terms of comparative plausibility, given in Figure 3.4 of this chapter.

$\mathbb{V}$		Axiomatization of classical propositional logic
	(CPR)	$\dfrac{\vdash B \to A}{\vdash A \preccurlyeq B}$
	(CPA)	$(A \preccurlyeq (A \lor B)) \lor (B \preccurlyeq (A \lor B))$
	(TR)	$(A \preccurlyeq B) \land (B \preccurlyeq C) \to (A \preccurlyeq C)$
	(CO)	$(A \preccurlyeq B) \lor (B \preccurlyeq A)$
Extensions of $\mathbb{V}$	(N)	$\neg(\bot \preccurlyeq \top)$
	(T)	$(\bot \preccurlyeq \neg A) \to A$
	(W)	$A \to (A \preccurlyeq \top)$
	(C)	$(A \preccurlyeq \top) \to A$

Proof of completeness with respect to the axiomatization of classical propositional logic employs only propositional rules of the calculus.

To derive (CPR), assume $\vdash B \to A$. This means that $\Rightarrow x : B \to A$. By substitution, also the sequent $\Rightarrow y : B \to A$ is derivable. By weakening, obtain $y : B \Rightarrow y : B \to A, y : A$. The sequent $y : B, y : B \to A \Rightarrow y : A$ is derivable. Apply the following rules:

$$\dfrac{\dfrac{\dfrac{\dfrac{y : B \Rightarrow y : A}{a \in N(x), y \in a, y : B \Rightarrow a \Vdash^{\exists} A, y : A}\;\mathsf{Wk}}{a \in N(x), y \in a, y : B \Rightarrow a \Vdash^{\exists} A}\;\mathsf{R}\Vdash^{\exists}}{a \in N(x), a \Vdash^{\exists} B \Rightarrow a \Vdash^{\exists} A}\;\mathsf{L}\Vdash^{\exists}}{\Rightarrow x : A \preccurlyeq B}\;\mathsf{R}\preccurlyeq$$

Thus, $\vdash A \to B$ we obtained $\vdash A \preccurlyeq B$, and we are done.

(CPR)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\triangledown}{\ldots y : A \Rightarrow \ldots y : A, y : B} \quad \overset{\triangledown}{\ldots y : B \Rightarrow \ldots y : A, y : B}}{\ldots y \in b, y \in a, a \subseteq b, y : A \lor B \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B, y : A, y : B}\;\mathsf{L}\lor}{\ldots y \in b, y \in a, a \subseteq b, y : A \lor B, \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\;\mathsf{R}\Vdash^{\exists}(2\mathsf{x})}{\ldots y \in a, a \subseteq b, y : A \lor B \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\;\mathsf{L}\subseteq}{a \subseteq b, a \in N(x), b \in N(x), a \Vdash^{\exists} A \lor B, b \Vdash^{\exists} A \lor B \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\;\mathsf{L}\Vdash^{\exists} \quad (*)}{a \in N(x), b \in N(x), a \Vdash^{\exists} A \lor B, b \Vdash^{\exists} A \lor B \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\;\mathsf{Nes}}{a \in N(x), a \Vdash^{\exists} A \lor B \Rightarrow a \Vdash^{\exists} A, x : B \preccurlyeq (A \lor B)}\;\mathsf{R}\preccurlyeq}{\Rightarrow x : A \preccurlyeq (A \lor B), x : B \preccurlyeq (A \lor B)}\;\mathsf{R}\preccurlyeq}{\Rightarrow x : (A \preccurlyeq (A \lor B)) \lor (B \preccurlyeq (A \lor B))}\;\mathsf{R}\lor$$

The rightmost premiss of Nes, sequent $(+)$ $b \subseteq a, a \in N(x), b \in N(x), a \Vdash^{\exists} A \lor B, b \Vdash^{\exists} A \lor B \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B$, is derived in a similar way.

(TR)

$$\dfrac{\dfrac{\dots a \Vdash^{\exists} \overset{\triangledown}{C} \Rightarrow \dots a \Vdash^{\exists} C \quad \dots a \Vdash^{\exists} \overset{\triangledown}{B} \Rightarrow \dots a \Vdash^{\exists} B}{a \in N(x), a \Vdash^{\exists} C, x : A \preccurlyeq B, x : B \preccurlyeq C \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B}\;\mathsf{L}\preccurlyeq \quad \dots a \Vdash^{\exists} \overset{\triangledown}{A} \Rightarrow \dots a \Vdash^{\exists} A}{\dfrac{a \in N(x), a \Vdash^{\exists} C, x : A \preccurlyeq B, x : B \preccurlyeq C \Rightarrow a \Vdash^{\exists} A}{\dfrac{x : A \preccurlyeq B, x : B \preccurlyeq C \Rightarrow x : A \preccurlyeq C}{\dfrac{x : (A \preccurlyeq B) \wedge (B \preccurlyeq C) \Rightarrow x : A \preccurlyeq C}{\Rightarrow x : (A \preccurlyeq B) \wedge (B \preccurlyeq C) \to (A \preccurlyeq C)}\;\mathsf{R}\to}\;\mathsf{L}\wedge}\;\mathsf{R}\preccurlyeq}\;\mathsf{L}\preccurlyeq$$

(CO) In the derivation the two premisses of Nes are denoted by (1) , sequent $a \subseteq b, a \in N(x), b \in N(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B$ and (2), sequent $b \subseteq a, a \in N(x), b \in N(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{a \subseteq b, y \in a, y \in b, \overset{\triangledown}{y} : B \cdots \Rightarrow \dots y : B}{a \subseteq b, y \in a, y \in b, y : B \cdots \Rightarrow a \Vdash^{\exists} B, b \Vdash^{\exists} A}\;\mathsf{R}\Vdash^{\exists}}{a \subseteq b, y \in a, y : B \cdots \Rightarrow a \Vdash^{\exists} B, b \Vdash^{\exists} A}\;\mathsf{L}\subseteq}{(1)}\;\mathsf{L}\Vdash^{\exists} \quad \dfrac{\dfrac{\dfrac{b \subseteq a, k \in b, k \in a, \overset{\triangledown}{k} : A \cdots \Rightarrow \dots k : A}{b \subseteq a, k \in b, k \in a, k : A \cdots \Rightarrow a \Vdash^{\exists} B, b \Vdash^{\exists} A}\;\mathsf{R}\Vdash^{\exists}}{b \subseteq a, k \in b, k : A \cdots \Rightarrow a \Vdash^{\exists} B, b \Vdash^{\exists} A}\;\mathsf{L}\subseteq}{(2)}\;\mathsf{L}\Vdash^{\exists}}{\dfrac{a \in N(x), b \in N(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}{\dfrac{a \in N(x), a \Vdash^{\exists} B, \Rightarrow x : B \preccurlyeq A, a \Vdash^{\exists} A}{\dfrac{\Rightarrow x : A \preccurlyeq B, x : B \preccurlyeq A}{\Rightarrow x : A \preccurlyeq B \vee B \preccurlyeq A}\;\mathsf{L}\vee}\;\mathsf{R}\preccurlyeq}\;\mathsf{R}\preccurlyeq}\;\mathsf{Nes}$$

(N)

$$\dfrac{\dfrac{\dfrac{y \in a, a \in N(x), x : \overset{\triangledown}{\bot} \preccurlyeq \top \Rightarrow a \Vdash^{\exists} \top, y : \top}{y \in a, a \in N(x), x : \bot \preccurlyeq \top \Rightarrow a \Vdash^{\exists} \top}\;\mathsf{R}\Vdash^{\exists}}{a \in N(x), x : \bot \preccurlyeq \top \Rightarrow a \Vdash^{\exists} \top}\;\mathsf{0} \quad \dfrac{z \in a, a \in N(x), \overset{\triangledown}{x} : \bot \preccurlyeq \top, z : \bot \Rightarrow}{a \in N(x), x : \bot \preccurlyeq \top, a \Vdash^{\exists} \bot \Rightarrow}\;\mathsf{L}\Vdash^{\exists}}{\dfrac{a \in N(x), x : \bot \preccurlyeq \top \Rightarrow}{\dfrac{x : \bot \preccurlyeq \top \Rightarrow}{\Rightarrow x : \neg(\bot \preccurlyeq \top)}\;\mathsf{R}\neg}\;\mathsf{N}}\;\mathsf{L}\preccurlyeq$$

(T)

$$\dfrac{\dfrac{\dfrac{\dots x \in a, x : \overset{\triangledown}{A} \Rightarrow x : A, a \Vdash^{\exists} \neg A}{\dots x \in a, a \in N(x) \Rightarrow x : A, a \Vdash^{\exists} \neg A, x : \neg A}\;\mathsf{R}\neg}{\dots x \in a, a \in N(x) \Rightarrow x : A, a \Vdash^{\exists} \neg A}\;\mathsf{R}\Vdash^{\exists} \quad \dfrac{\dots y \in a, y : \bot, \overset{\triangledown}{x} : \bot \preccurlyeq \neg A \Rightarrow x : A}{\dots a \Vdash^{\exists} \bot, x : \bot \preccurlyeq \neg A \Rightarrow x : A}\;\mathsf{R}\Vdash^{\exists}}{\dfrac{x \in a, a \in N(x), x : \bot \preccurlyeq \neg A \Rightarrow x : A}{\dfrac{x : \bot \preccurlyeq \neg A \Rightarrow x : A}{\Rightarrow x : (\bot \preccurlyeq \neg A) \to A}\;\mathsf{R}\to}\;\mathsf{T}}\;\mathsf{L}\preccurlyeq$$

(W)

$$\dfrac{\dfrac{\dfrac{\dfrac{x \in a, a \in N(x), a \Vdash^{\exists} \overset{\triangledown}{\top}, x : A \Rightarrow a \Vdash^{\exists} A, x : A}{x \in a, a \in N(x), a \Vdash^{\exists} \top, x : A \Rightarrow a \Vdash^{\exists} A}\;\mathsf{R}\Vdash^{\exists}}{a \in N(x), a \Vdash^{\exists} \top, x : A \Rightarrow a \Vdash^{\exists} A}\;\mathsf{W}}{x : A \Rightarrow x : A \preccurlyeq \top}\;\mathsf{R}\preccurlyeq}{\Rightarrow x : A \to (A \preccurlyeq \top)}\;\mathsf{R}\to$$

(C)

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\overset{\triangledown}{\dots x \in \{x\} \Rightarrow x : A, \{x\} \Vdash^{\exists} \top, x : \top}}{\dots x \in \{x\}, \{x\} \in N(x) \Rightarrow x : A, \{x\} \Vdash^{\exists} \top}\,\mathsf{R} \Vdash^{\exists}
}{\dots \{x\} \in N(x) \Rightarrow x : A, \{x\} \Vdash^{\exists} \top}\,\mathsf{Single}
\qquad
\dfrac{
\dfrac{\overset{\triangledown}{y \in \{x\}, y : A, x : A \cdots \Rightarrow x : A}}{\dots y \in \{x\}, y : A, \{x\} \in N(x) \Rightarrow x : A}\,\mathsf{Repl}_1
}{\dots \{x\} \Vdash^{\exists} A, \{x\} \in N(x) \Rightarrow x : A}\,\mathsf{L} \Vdash^{\exists}
}{
\dfrac{\{x\} \in N(x), a \in N(x), x : A \preccurlyeq \top \Rightarrow x : A}{
\dfrac{a \in N(x), x : A \preccurlyeq \top \Rightarrow x : A}{
\dfrac{x : A \preccurlyeq \top \Rightarrow x : A}{\Rightarrow x : (A \preccurlyeq \top) \to A}\,\mathsf{R} \to
}\,\mathsf{N}
}\,\mathsf{C}
}\,\mathsf{L} \preccurlyeq
}{}
$$

Proof of Theorem 3.8.6. Derivation of the axioms and admissibility of the inference rules of $\mathbb{PCL}$ for sequent calculus $\mathbf{G3A_w}$ and extensions are proved similarly as for calculus **G3CL** (refer to proof of Theorem 3.4.7 at the beginning of this appendix). We show derivations of the absoluteness axioms in $\mathbf{G3A_w}$ and derivations of the axioms for the extensions of $\mathbb{PCLA}$ in the corresponding calculi $\mathbf{G3A^*_w}$. The axioms are given in proof of Theorem 3.4.7 at the beginning of this appendix.

(A_1)

$$
\dfrac{
(*) \qquad
\dfrac{
\dfrac{
\dfrac{\overset{\triangledown}{a \subseteq a, y : C, y \in a, a \Vdash^{\exists} C, w : A > B \Rightarrow_w a : C|A > B, w : A > B}}{a \subseteq a, y : C, y \in a, a \Vdash^{\exists} C, w : A > B \Rightarrow_w a : C|A > B, y : A > B}\,\mathsf{LA}
}{a \subseteq a, y \in a, a \Vdash^{\exists} C, w : A > B \Rightarrow_w a : C|A > B, y : C \to (A > B)}\,\mathsf{R} \to_{\mathsf{A}}
}{a \subseteq a, a \Vdash^{\exists} C, w : A > B \Rightarrow_w a : C|A > B, a \Vdash^{\forall} C \to (A > B)}\,\mathsf{R} \Vdash^{\forall}_{\mathsf{A}}
}{
\dfrac{a \subseteq a, a \Vdash^{\exists} C, w : A > B \Rightarrow_w a : C|A > B}{
\dfrac{a \Vdash^{\exists} C, w : A > B \Rightarrow_w a : C|A > B}{w : A > B \Rightarrow_w w : C > (A > B)}\,\mathsf{R} >_{\mathsf{A}}
}\,\mathsf{Ref}_{\mathsf{A}}
}\,\mathsf{R}|_{\mathsf{A}}
$$

Sequent $(*)$, the leftmost premiss of $\mathsf{R}|_{\mathsf{A}}$, is the derivable sequent $a \subseteq a, a \Vdash^{\exists} C, w : A > B \Rightarrow_w a : C|A > B, a \Vdash^{\exists} C$.

(A_2)

$$
\dfrac{
(*) \qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\overset{\triangledown}{y \in a, a \subseteq a, a \Vdash^{\exists} C, y : C, w : A > B \Rightarrow_w w : A > B, w : C > \neg(A > B), a : C|\neg(A > B)}}{y \in a, a \subseteq a, a \Vdash^{\exists} C, y : C, y : A > B \Rightarrow_w w : A > B, w : C > \neg(A > B), a : C|\neg(A > B)}\,\mathsf{RA}
}{y \in a, a \subseteq a, a \Vdash^{\exists} C, y : C \Rightarrow_w w : A > B, w : C > \neg(A > B), a : C|\neg(A > B), y : \neg(A > B)}\,\mathsf{R}\neg_{\mathsf{A}}
}{y \in a, a \subseteq a, a \Vdash^{\exists} C \Rightarrow_w w : A > B, w : C > \neg(A > B), a : C|\neg(A > B), y : C \to \neg(A > B)}\,\mathsf{R} \to_{\mathsf{A}}
}{a \subseteq a, a \Vdash^{\exists} C \Rightarrow_w w : A > B, w : C > \neg(A > B), a : C|\neg(A > B), a \Vdash^{\forall} C \to \neg(A > B)}\,\mathsf{R} \Vdash^{\forall}_{\mathsf{A}}
}{
\dfrac{a \subseteq a, a \Vdash^{\exists} C \Rightarrow_w w : A > B, w : C > \neg(A > B), a : C|\neg(A > B)}{
\dfrac{a \Vdash^{\exists} C \Rightarrow_w w : A > B, w : C > \neg(A > B), a : C|\neg(A > B)}{
\dfrac{\Rightarrow_w w : A > B, w : C > \neg(A > B)}{
\dfrac{w : \neg(A > B) \Rightarrow_w w : C > \neg(A > B)}{\Rightarrow_w w : \neg(A > B) \to (C > \neg(A > B))}\,\mathsf{R} \to_{\mathsf{A}}
}\,\mathsf{L}\neg_{\mathsf{A}}
}\,\mathsf{R} >_{\mathsf{A}}
}\,\mathsf{Ref}_{\mathsf{A}}
}\,\mathsf{R}|_{\mathsf{A}}
$$

Sequent $(*)$, the leftmost premiss of $\mathsf{R}|_{\mathsf{A}}$, is the derivable sequent $a \subseteq a, a \Vdash^{\exists} C \Rightarrow_w w : A > B, w : C > \neg(A > B), a : C|\neg(A > B), a \Vdash^{\exists} C$.

(N)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{y \in a, a \subseteq a, w : \top > \overset{\triangledown}{\bot} \Rightarrow_w a \Vdash^{\exists} \top, y : \top}}{y \in a, a \subseteq a, w : \top > \bot \Rightarrow_w a \Vdash^{\exists} \top}\ \mathsf{R}\Vdash^{\forall}}{a \subseteq a, w : \top > \bot \Rightarrow_w a \Vdash^{\exists} \top}\ \mathsf{0_A}}{}}{}}{}}{}}{}}{}}{}}{}}{}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{z : \bot \cdots \Rightarrow_w \quad \cdots \Rightarrow_w z : \top}{z \in c, c \subseteq a, z : \top, c \Vdash^{\forall} \top \to \bot, z : \top \to \bot, a \subseteq a, w : \top > \bot \Rightarrow_w}\ \mathsf{L} \to_{\mathsf{A}}}{z \in c, c \subseteq a, z : \top, c \Vdash^{\forall} \top \to \bot, a \subseteq a, w : \top > \bot \Rightarrow_w}\ \mathsf{L} \Vdash^{\forall}_{\mathsf{A}}}{c \subseteq a, c \Vdash^{\exists} \top, c \Vdash^{\forall} \top \to \bot, a \subseteq a, w : \top > \bot \Rightarrow_w}\ \mathsf{L} \Vdash^{\exists}_{\mathsf{A}}}{a : \top | \bot, a \subseteq a, w : \top > \bot \Rightarrow_w}\ \mathsf{L}|_{\mathsf{A}}
$$

$$
\dfrac{\dfrac{\dfrac{a \subseteq a, w : \top > \bot \Rightarrow_w a \Vdash^{\exists} \top \qquad a : \top | \bot, a \subseteq a, w : \top > \bot \Rightarrow_w}{a \subseteq a, w : \top > \bot \Rightarrow_w}\ \mathsf{L} >_{\mathsf{A}}}{w : \top > \bot \Rightarrow_w}\ \mathsf{N_A}}{\Rightarrow_w w : \neg(\top > \bot)}\ \mathsf{L}\neg
$$

(T)

$$
\dfrac{\dfrac{\dfrac{\dfrac{w : \overset{\triangledown}{\bot} \cdots \Rightarrow_w \quad w : A \cdots \overset{\triangledown}{\Rightarrow_w} w : A}{b \subseteq a, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to \bot, w : A \to \bot, w \in a, w : A, w : A > \bot \Rightarrow_w}\ \mathsf{L} \to_{\mathsf{A}}}{b \subseteq a, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to \bot, w \in a, w : A, w : A > \bot \Rightarrow_w}\ \mathsf{L} \Vdash^{\forall}_{\mathsf{A}}}{a : A | \bot, x \in a, w : A, w : A > \bot \Rightarrow_w}\ \mathsf{L}|_{\mathsf{A}}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{w \in a, w : A, w : \overset{\triangledown}{A} > \bot \Rightarrow_w a \Vdash^{\exists} A \qquad a : A | \bot, x \in a, w : A, w : A > \bot \Rightarrow_w}{w \in a, w : A, w : A > \bot \Rightarrow_w}\ \mathsf{L} >_{\mathsf{A}}}{w : A, w : A > \bot \Rightarrow_w}\ \mathsf{T_A}}{w : A \Rightarrow_w w : \neg(A > \bot)}\ \mathsf{L}\neg}{\Rightarrow_w w : A \to \neg(A > \bot)}\ \mathsf{R} \to_{\mathsf{A}}
$$

(W)

$$
\dfrac{
\dfrac{w \in a, a \subseteq a, w : A > \overset{\triangledown}{B}, w : A \Rightarrow_w w : B, a \Vdash^{\exists} a}{a \subseteq a, w : A > B, w : A \Rightarrow_w w : B, a \Vdash^{\exists} a}\;\mathsf{W_A}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{w : B \cdots \overset{\triangledown}{\Rightarrow_w} w : B \quad \ldots w : A \overset{\triangledown}{\cdots} \Rightarrow_w w : A}{w \in b, b \subseteq a, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, w : A \to B, a \subseteq a, w : A > B, w : A \Rightarrow_w w : B}\;\mathsf{L} \to_{\mathsf{A}}}{w \in b, b \subseteq a, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, a \subseteq a, w : A > B, w : A \Rightarrow_w w : B}\;\mathsf{R} \Vdash^{\forall}_{\mathsf{A}}}{b \subseteq a, b \Vdash^{\exists} A, b \Vdash^{\forall} A \to B, a \subseteq a, w : A > B, w : A \Rightarrow_w w : B}\;\mathsf{W_A}}{a : A|B, a \subseteq a, w : A > B, w : A \Rightarrow_w w : B}\;\mathsf{L}|_{\mathsf{A}}
}{
\dfrac{\dfrac{\dfrac{a \subseteq a, w : A > B, w : A \Rightarrow_w w : B}{w : A > B, w : A \Rightarrow_w w : B}\;\mathsf{N_A}}{w : A > B \Rightarrow_w w : A \to B}\;\mathsf{R} \to}{\Rightarrow_w w : (A > B) \to (A \to B)}\;\mathsf{R} \to_{\mathsf{A}}
}\;\mathsf{L} >_{\mathsf{A}}
$$

(C)

$$
\dfrac{
\dfrac{\dfrac{\dfrac{w \in a, w \in \{w\}, \{w\} \subseteq a, a \overset{\triangledown}{\Vdash^{\exists}} A, w : A, w : B \Rightarrow_w \ldots w : A}{w \in a, w \in \{w\}, \{w\} \subseteq a, a \Vdash^{\exists} A, w : A, w : B \Rightarrow_w \ldots \{w\} \Vdash^{\exists} A}\;\mathsf{R} \Vdash^{\exists}_{\mathsf{A}}}{w \in a, w \in \{w\}, \{w\} \subseteq a, a \Vdash^{\exists} A, w : A, w : B \Rightarrow_w \ldots \{w\} \Vdash^{\exists} A}\;\mathsf{L} \subseteq_{\mathsf{A}}}{\{w\} \subseteq a, a \Vdash^{\exists} A, w : A, w : B \Rightarrow_w \ldots \{w\} \Vdash^{\exists} A}\;\mathsf{Single_A}
\qquad
\dfrac{\dfrac{\dfrac{y \in \{w\}, \{w\} \subseteq a, a \Vdash^{\exists} A, w : A, \overset{\triangledown}{w} : B, y : B, y : A \Rightarrow_w a : A|B, y : B}{y \in \{w\}, \{w\} \subseteq a, a \Vdash^{\exists} A, w : A, w : B, y : A \Rightarrow_w a : A|B, y : B}\;\mathsf{Repl_{A1}}}{y \in \{w\}, \{w\} \subseteq a, a \Vdash^{\exists} A, w : A, w : B \Rightarrow_w a : A|B, y : A \to B}\;\mathsf{R} \to_{\mathsf{A}}}{\{w\} \subseteq a, a \Vdash^{\exists} A, w : A, w : B \Rightarrow_w a : A|B, \{w\} \Vdash^{\forall} A \to B}\;\mathsf{R} \Vdash^{\forall}_{\mathsf{A}}
}{
\dfrac{\dfrac{\dfrac{\dfrac{\{w\} \subseteq a, a \Vdash^{\exists} A, w : A, w : B \Rightarrow_w a : A|B}{a \Vdash^{\exists} A, w : A, w : B \Rightarrow_w a : A|B}\;\mathsf{C_A}}{w : A, w : B \Rightarrow_w w : A > B}\;\mathsf{R} >_{\mathsf{A}}}{w : A \wedge B \Rightarrow_w w : A > B}\;\mathsf{L}\wedge}{\Rightarrow_w w : (A \wedge B) \to (A > B)}\;\mathsf{R} \to_{\mathsf{A}}
}\;\mathsf{R}|_{\mathsf{A}}
$$

Chapter 4

Internal calculi for Lewis' conditional logics

Reality is frequently inaccurate.
– Douglas Adams,
The Restaurant at the End of the Universe

This chapter considers conditional logics belonging to Lewis' family. Neighbourhood models for these logics have the property of *nesting*: given a world x and any two neighbourhoods $\alpha, \beta \in N(x)$, either α is included in β or β is included in α.
We present three families of proof systems, covering different sub-families of Lewis' logics. The calculi are *internal*, meaning that, instead of enriching the language, they add structural connectives to the formalism of Gentzen's sequent calculus. We introduce a family of one-level nested sequent calculi $\mathcal{I}^{i}_{\mathsf{Base}}$ for $\mathbb{V}$, $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$, $\mathbb{VC}$, $\mathbb{VA}$ and $\mathbb{VNA}$; a family of extended hypersequent calculi $\mathcal{SH}^{i}_{\mathsf{Strong}}$ for $\mathbb{VTU}$ (logic $\mathbb{V}$ with total reflexivity and uniformity) and its extensions; and finally, a family of head-hypersequent calculi $\mathcal{HH}^{i}_{\mathsf{Strong}^{\mathsf{A}}}$ for $\mathbb{VTA}$ and its extensions. This latter proof system is obtained by means of an additional modification on the formalism of hypersequent, and the resulting proof systems are suitable to capturing logics with absoluteness and total reflexivity.

4.1 Lewis' family of conditional logics

The main goal of Lewis' seminal work *Counterfactuals* was to define a formal system capturing counterfactual conditionals. As seen in Chapter 1, he defined a whole family of conditional logics, $\mathbb{V}$ and its extensions, and identified $\mathbb{VC}$ as the logic of counterfactuals.

The common feature of conditional logics in Lewis' family is that they can all be characterized by means of *nested* neighbourhood models or - as Lewis called them - sphere models. According to Lewis, the condition of nesting is crucial when defining a logic for counterfactuals, since it ensures that the worlds in the model are always comparable with one another in terms of similarity with the actual world: words belonging to the inner spheres are more similar to the actual world with respect to worlds belonging to outer spheres.

The condition of nesting is also crucial for the definition of *internal* proof systems for conditional logics. As done in Section 3.7 when dealing with labelled proof systems for

$\mathbb{V}$ logics, we take as primitive the comparative plausibility operator[1]. In neighbourhood models with nesting, the comparative plausibility operator $\preccurlyeq$ has a remarkably simple truth condition, which allows for an elegant treatment in the internal proof system. The internal calculi presented in this chapter enrich the structure of the sequent by means of a block structure corresponding to a disjunction of comparative plausibility formulas.

$$[A_1, \dots, A_n \lhd C] := A_1 \preccurlyeq C \vee \dots \vee A_n \preccurlyeq C$$

The calculi were first introduced in [80], and can be found in [35, 37].

Remark 4.1.1. In this chapter we present only *standard* proof systems: calculi with a finite number of rules, and in which every rule has a fixed and finite number of premisses. However, *non-standard* calculi for Lewis' conditional logics with strong structural properties do exist: they have been defined by Lellmann and Pattinson in [53]. In [35] and [37] the non-standard systems are presented, along with their relationship with the standard calculi.

We recall the language and the classes of models for Lewis' conditional logics, given in the previous chapter (Definition 3.7.1 and Definition 3.1.3).

Definition 4.1.1. The set of well- formed formulas of $\mathbb{V}$ and its extensions is defined by means of the following grammar, for $p \in \mathcal{V}$ propositional variable and $A \in \mathcal{F}^{\preccurlyeq}$:

$$\mathcal{F}^{\preccurlyeq} ::= p \mid \bot \mid A \to A \mid A \preccurlyeq A.$$

Definition 4.1.2. A *nested neighbourhood model* (or a *sphere model*) is a structure

$$\mathcal{M} = \langle W, N, [\![\ ]\!] \rangle$$

composed of a non-empty set of elements, W; a function $N : W \to \mathcal{P}(\mathcal{P}(W))$, associating to each world x a set of sets of worlds $N(x)$ and a propositional evaluation $[\![\ ]\!] : \mathcal{V} \to \mathcal{P}(W)$. In this chapter, we call "spheres" the elements of $N(x)$. Moreover, N satisfies the following properties:

- *Non-emptiness*: For each $\alpha \in N(x)$, α is non-empty;
- *Nesting*: For each $\alpha, \beta \in N(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

The valuation $[\![\ ]\!]$ is routinely extended to all formulas by:

- $[\![\bot]\!] = \emptyset$;
- $[\![A \to B]\!] = (W - [\![A]\!]) \cup [\![B]\!]$;
- $[\![A \preccurlyeq B]\!] = \{w \in W \mid \text{for all } \alpha \in N(w), \text{ if } [\![B]\!] \cap \alpha \neq \emptyset, \text{ then } [\![A]\!] \cap \alpha \neq \emptyset\}$.

Then, adopting the notation $w \Vdash A$ for $w \in [\![A]\!]$ and the forcing relations $\alpha \Vdash^{\forall} A$ (for all $x \in \alpha$ it holds that $x \Vdash A$) and $\alpha \Vdash^{\exists} A$ (there exists $x \in \alpha$ such that $x \Vdash A$), satisfiability of a $\preccurlyeq$-formula in a model becomes the following:

$$x \Vdash A \preccurlyeq B \quad \textit{iff} \quad \textit{for all } \alpha \in N(x) \textit{ it holds that if } \alpha \Vdash^{\exists} B \textit{ then } \alpha \Vdash^{\exists} A.$$

Conditional logic $\mathbb{V}$ is the set of formulas valid in all sphere models (proof: [57]). As mentioned in the previous chapters, extensions of $\mathbb{V}$ are semantically defined by specifying additional conditions on the class of sphere models, namely:

N *Normality*: For all $x \in W$ it holds that $N(x) \neq \emptyset$;

[1] Recall that the conditional operator $>$ and the comparative plausibility operator are interdefinable by means of the following formulas: $A > B \equiv (\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \wedge B))$ and $A \preccurlyeq B \equiv ((A \vee B) > \bot) \vee \neg((A \vee B) > \neg A)$.

- T *Total reflexivity*: For all $x \in W$ there exists $\alpha \in N(x)$ such that $x \in \alpha$;
- W *Weak centering*: Normality holds, and for all $x \in W$, for all $\alpha \in N(x)$ it holds that $x \in \alpha$;
- C *Centering*: For all $x \in W$, for all $\alpha \in N(x)$ it holds that $\{x\} \in N(x)$ so that $\{x\} \subseteq \alpha$;
- U *Uniformity*: For all $x, y \in W$ it holds that $\bigcup N(x) = \bigcup N(y)$;
- A *Absoluteness*: For all $x, y \in W$ it holds that $N(x) = N(y)$[2].

Extensions of $\mathbb{V}$ are denoted by concatenating the letters for these properties: $\mathbb{N}$ for normality, $\mathbb{T}$ for total reflexivity, $\mathbb{W}$ for weak centering, $\mathbb{C}$ for centering, and $\mathbb{A}$ for absoluteness. Figure 4.1 presents an axiomatization of the family of logics in terms of $\preccurlyeq$, from [57, Chp. 6][3]. We assume $\vee$ and $\wedge$ bind stronger than $\preccurlyeq$.

$\mathbb{V}$		Axiomatization of classical propositional logic
	(CPR)	$\dfrac{\vdash B \to A}{\vdash A \preccurlyeq B}$
	(CPA)	$(A \preccurlyeq A \vee B) \vee (B \preccurlyeq A \vee B)$
	(TR)	$(A \preccurlyeq B) \wedge (B \preccurlyeq C) \to (A \preccurlyeq C)$
	(CO)	$(A \preccurlyeq B) \vee (B \preccurlyeq A)$

Extensions		Axiomatization of $\mathbb{V}$
of $\mathbb{V}$	(N)	$\neg(\bot \preccurlyeq \top)$
	(T)	$(\bot \preccurlyeq \neg A) \to A$
	(W)	$A \to (A \preccurlyeq \top)$
	(C)	$(A \preccurlyeq \top) \to A$
	(U_1)	$\neg(\bot \preccurlyeq A) \to \big(\bot \preccurlyeq (\bot \preccurlyeq A)\big)$
	(U_2)	$(\bot \preccurlyeq \neg A) \to \big(\bot \preccurlyeq \neg(\bot \preccurlyeq \neg A)\big)$
	(A_1)	$\neg(A \preccurlyeq B) \to \big(\bot \preccurlyeq (A \preccurlyeq B)\big)$
	(A_2)	$(A \preccurlyeq B) \to \big(\bot \preccurlyeq \neg(A \preccurlyeq B)\big)$

$$\mathcal{A}_{\mathbb{V}} = \{\mathsf{CPR}, \mathsf{CPA}, \mathsf{TR}, \mathsf{CO}\}$$

$$\begin{array}{llllll}
\mathcal{A}_{\mathbb{VN}} = & \mathcal{A}_{\mathbb{V}} \cup \{\mathsf{N}\} & \mathcal{A}_{\mathbb{VT}} = & \mathcal{A}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}\} & \mathcal{A}_{\mathbb{VW}} = & \mathcal{A}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}, \mathsf{W}\} \\
\mathcal{A}_{\mathbb{VC}} = & \mathcal{A}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}, \mathsf{W}, \mathsf{C}\} & \mathcal{A}_{\mathbb{VA=}} & \mathcal{A}_{\mathbb{V}} \cup \{\mathsf{A1}, \mathsf{A2}\} & \mathcal{A}_{\mathbb{VNA=}} & \mathcal{A}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{A1}, \mathsf{A2}\}
\end{array}$$

Figure 4.1: Lewis' logics and axioms.

The two modal operators $\Box$ and $\Diamond$ can be defined in terms of $\preccurlyeq$ as follows:

$$\Box A = \bot \preccurlyeq \neg A$$
$$\Diamond A = \neg(\bot \preccurlyeq A).$$

Not unexpectedly, the correspondence between modal and conditional logics described in Chapter 1 in terms of $>$ continues to hold also when considering $\preccurlyeq$ as primary operator. Thus, for instance, Axiom (T) corresponds to the axiom of reflexivity in modal logics, and axioms (U_1) and (U_2) correspond respectively to axioms (5) and axiom (4) of modal logics.

[2]In this case we are using the global versions of the rules of uniformity and absoluteness. Models with the local or global versions of the rules satisfy the same sets of formulas.

[3]This is the complete axiomatization: in Figure 3.4 of Chapter 3 we presented only some of the axioms for the extensions.

Lewis' conditional logic	*Outer modal logic*
$\mathbb{V}$	**K**
$\mathbb{VN}$	**D**
$\mathbb{VT}$, $\mathbb{VW}$, $\mathbb{VC}$	**T**
$\mathbb{VU}$, $\mathbb{VA}$	**K45**
$\mathbb{VNU}$, $\mathbb{VNA}$	**D45**
$\mathbb{VTU}$, $\mathbb{VWU}$, $\mathbb{VCU}$, $\mathbb{VTA}$, $\mathbb{VWA}$	**S5**
$\mathbb{VCA}$	Classical logic

Figure 4.2: Modal correspondences

To fully understand the internal calculi we are going to present in the following sections, it is important to keep in mind the relations between modal and conditional logics, detailed in Section 1.6. Basically, the proof systems we are going to define for conditional logics are similar to the proof systems for the corresponding outer modal logics. Thus, we are going to define a kind of nested sequent calculus to treat the weaker systems of Lewis' logics (corresponding to systems whose outer modal logic is weaker than **S5**) while for $\mathbb{VTU}$ and extensions (whose outer modal logic is **S5**) we introduce a hypersequent calculus. We do not present sequent calculi for $\mathbb{VU}$ and $\mathbb{VNU}$; some remarks on these systems can be found in Section 4.11.

4.2 Nested sequent calculi for basic systems

In this section we present internal calculi for the basic Lewis' logic $\mathbb{V}$ as well as for some of its extensions. We shall consider $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$, $\mathbb{VC}$, $\mathbb{VA}$ and $\mathbb{VNA}$. For notational convenience in the following we take Base to range over the set $\{\mathbb{V}, \mathbb{VN}, \mathbb{VT}, \mathbb{VW}, \mathbb{VC}, \mathbb{VA}, \mathbb{VNA}\}$. We first present $\mathcal{I}_{\mathsf{Base}}$, a non-invertible family of sequent calculi, and then $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$, its invertible version. The former is needed to prove cut elimination, the second to prove semantic completeness.

Definition 4.2.1. A *conditional block* is a syntactic structure consisting of a multiset Σ of formulas and a single formula A, written $[\Sigma \lhd A]$. A *sequent* is a tuple $\Gamma \Rightarrow \Delta$, where Γ is a multiset of formulas, and Δ is a multiset of formulas and conditional blocks. The *formula interpretation* of a sequent is given by (all blocks shown):

$$(\Gamma \Rightarrow \Delta', [\Sigma_1 \lhd C_1], \dots, [\Sigma_n \lhd C_n])^{int} := \bigwedge \Gamma \to \bigvee \Delta' \vee \bigvee_{1 \le i \le n} \bigvee_{B \in \Sigma_i} (B \preccurlyeq C_i)$$

Figure 4.3 presents non-invertible calculi for systems Base. We write $[\Theta, \Sigma \lhd A]$ for $[(\Theta, \Sigma) \lhd A]$, with Θ, Σ standing for multiset union. Propositional rules and initial sequents are standard. Rules for contraction both on the sequent level and inside blocks are included for technical convenience, but they are not needed for completeness. Rule $R \preccurlyeq$ introduces a block structure in the premiss, and rule $L \preccurlyeq$ shows how a comparative plausibility formula in the leftmost side of a sequent can be included into an existing block.

Intuitively, each block can be read as encoding the comparative plausibility formulas relative to a sphere in the model. Rule com corresponds to the nesting of neighbourhoods, which allows for the comparative plausibility formulas to "pass" between spheres. The jump rule, considered bottom-up, allows to go from the block level to the formula level, "jumping" to a new world of the model[4]. Thanks to this rule, the level of nesting in sequent calculi $\mathcal{I}_{\mathsf{Base}}$ remains at most one. Differently to what happens with nested sequents, we cannot introduce a block structure inside a block. The structure of sequents is simpler than in a full nested sequent calculus; however, this operation has a price: the jump rule is not invertible, and it erases all formulas not belonging to the block. The advantage is that the level of nesting - and thus, the complexity of the calculus - is kept at his minimum.

Initial sequents

$$\frac{}{\Gamma, \bot \Rightarrow \Delta}\ \bot_{\mathsf{L}} \qquad \frac{}{\Gamma, p \Rightarrow \Delta, p}\ \mathsf{init}$$

Propositional and conditional rules

$$\frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \rightarrow B \Rightarrow \Delta}\ \mathsf{L}\rightarrow \qquad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ \mathsf{R}\rightarrow$$

$$\frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B}\ R\preccurlyeq \qquad \frac{A \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{jump}$$

$$\frac{\Gamma \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma \Rightarrow \Delta, [\Sigma \lhd A]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C]}\ L\preccurlyeq$$

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A] \quad \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}\ \mathsf{com}$$

Structural rules

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \mathsf{Ctr_L} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \mathsf{Ctr_R}$$

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd A]}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{Ctr_B} \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A, A \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma, A \lhd B]}\ \mathsf{Ctr_i}$$

Rules for extensions

$$\frac{\Gamma \Rightarrow \Delta, [\bot \lhd \top]}{\Gamma \Rightarrow \Delta}\ \mathsf{N} \qquad \frac{\Gamma \Rightarrow \Delta, B \qquad \Gamma \Rightarrow \Delta, [\bot \lhd A]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta}\ \mathsf{T}$$

$$\frac{\Gamma \Rightarrow \Delta, \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{W} \quad \frac{\Gamma, C \Rightarrow \Delta \qquad \Gamma \Rightarrow D, \Delta}{\Gamma, C \preccurlyeq D \Rightarrow \Delta}\ \mathsf{C} \quad \frac{\Gamma^{\preccurlyeq}, B \Rightarrow \Delta^{\preccurlyeq}, \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd B]}\ \mathsf{A}$$

$\Gamma^{\preccurlyeq} \Rightarrow \Delta^{\preccurlyeq}$ is $\Gamma \Rightarrow \Delta$ restricted to formulas of the form $C \preccurlyeq D$ and blocks.

$$\mathcal{I}_{\mathbb{V}} := \{\bot, \mathsf{init}, \mathsf{L}\rightarrow, \mathsf{R}\rightarrow, R\preccurlyeq, L\preccurlyeq, \mathsf{com}, \mathsf{jump}, \mathsf{Ctr_R}, \mathsf{Ctr_L}, \mathsf{Ctr_B}\}$$

$\mathcal{I}_{\mathbb{VN}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}\}$ $\quad$ $\mathcal{I}_{\mathbb{VW}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}, \mathsf{W}\}$ $\quad$ $\mathcal{I}_{\mathbb{VA}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{A}\}$
$\mathcal{I}_{\mathbb{VT}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}\}$ $\quad$ $\mathcal{I}_{\mathbb{VC}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{TW}, \mathsf{C}\}$ $\quad$ $\mathcal{I}_{\mathbb{VNA}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{A}\}$

Figure 4.3: The calculus $\mathcal{I}_{\mathbb{V}}$ and its extensions

Example 4.2.1. To illustrate the use of the calculus we show a derivation of the characteristic axiom $(\bot \preccurlyeq \neg A) \rightarrow A$ for logic $\mathbb{VT}$ in the calculus $\mathcal{I}_{\mathbb{VW}}$ and a derivation of it in the calculus $\mathbb{VC}$ (where $\neg A = (A \rightarrow \bot)$):

[4]Refer to Chapter 5 for some more formal observations on the correspondence between formulas and blocks.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{\triangledown}{A \Rightarrow A, \bot, \bot}}{\Rightarrow A, A \to \bot, \bot}\ \mathsf{R}\to}{\Rightarrow A, [(A \to \bot), \bot \lhd \top]}\ \mathsf{W} \qquad \dfrac{\overset{\triangledown}{\bot \Rightarrow \bot}}{\Rightarrow A, [\bot \lhd \bot]}\ \mathsf{jump}}{\bot \preccurlyeq (A \to \bot) \Rightarrow A, [\bot \lhd \top]}\ L\preccurlyeq}{\dfrac{\bot \preccurlyeq (A \to \bot) \Rightarrow A}{\Rightarrow (\bot \preccurlyeq (A \to \bot)) \to A}\ \mathsf{R}\to}\ \mathsf{N}
\qquad
\dfrac{\overset{\triangledown}{\bot \Rightarrow A} \quad \dfrac{\overset{\triangledown}{A \Rightarrow A, \bot}}{\Rightarrow A, A \to \bot}\ \mathsf{R}\to}{\bot \preccurlyeq (A \to \bot) \Rightarrow A}\ \mathsf{C}
$$

Therefore, rule T could be omitted in the rule sets $\mathcal{I}_{\mathbb{VW}}$ and $\mathcal{I}_{\mathbb{VC}}$.

Remark 4.2.1. Given the definition of $>$ in terms of $\preccurlyeq$, rules for counterfactual implication can be explicitly stated as follows:

$$
\dfrac{\bot \preccurlyeq A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, [A \wedge \neg B \lhd A]}{A > B, \Gamma \Rightarrow \Delta}\ >_{\mathsf{L}} \qquad \dfrac{(A \wedge \neg B) \preccurlyeq A, \Gamma \Rightarrow \Delta, [\bot \lhd A]}{\Gamma \Rightarrow \Delta, A > B}\ >_{\mathsf{R}}
$$

The soundness proof might help clarifying the operational meaning of the sequent calculus rules. Completeness is postponed to section 4.5.

Theorem 4.2.1 (Soundness)**.** If $\mathcal{I}_{\mathsf{Base}} \vdash \Gamma \Rightarrow \Delta$, then $(\Gamma \Rightarrow \Delta)^{int}$ is a theorem of Base.

Proof. We prove that all the rules preserve soundness with respect to the formula interpretation. In particular, for every rule of the calculus $\mathcal{I}_{\mathsf{Base}}$ we show that if there is a countermodel in the class of models for Base for the formula interpretation of its conclusion, then there also is such a countermodel for at least one of its premisses. The cases of the propositional rules and initial sequents are standard.

$L \preccurlyeq$: Suppose we have a model $\langle W, N, [\![\,]\!]\rangle$ and a world $w \in W$ such that $w \nVdash \bigwedge \Gamma \wedge C \preccurlyeq D \to \bigvee \Delta \vee \bigvee_{B \in \Sigma} B \preccurlyeq A$. Then $w \Vdash \bigwedge \Gamma \wedge C \preccurlyeq D \wedge \neg \bigvee \Delta \wedge \bigwedge_{B \in \Sigma} \neg B \preccurlyeq A$. If $w \nVdash D \preccurlyeq A$, then $w \nVdash \bigwedge \Gamma \to \bigvee \Delta \vee \bigvee_{B \in \Sigma} B \preccurlyeq A \vee D \preccurlyeq A$, i.e. the formula interpretation of the first premiss of $L \preccurlyeq$ is not satisfied. Otherwise we have $w \Vdash D \preccurlyeq A$ and hence by transitivity and the fact that $w \Vdash C \preccurlyeq D$ also $w \Vdash C \preccurlyeq A$. But then for every $B \in \Sigma$ we have $w \Vdash \neg B \preccurlyeq C$, since otherwise by transitivity we would have $w \Vdash B \preccurlyeq A$. Hence the formula interpretation of the second premiss does not hold in w.

$R \preccurlyeq$: Immediate from the formula interpretation of blocks.

com: Suppose that for a model $\langle W, N, [\![\,]\!]\rangle$ and $w \in W$ we have $w \nVdash \bigwedge \Gamma \to \bigvee \Delta \vee \bigvee_{C \in \Sigma_1} C \preccurlyeq A \vee \bigvee_{D \in \Sigma_2} D \preccurlyeq B$. Since the axiom (CO) is valid in the model we have $w \Vdash A \preccurlyeq B$ or $w \Vdash B \preccurlyeq A$. In the first case by transitivity we have that $w \nVdash C \preccurlyeq B$ for every $C \in \Sigma_1$, and hence the formula interpretation of the first premiss of com does not hold in w. In the second case we have that $w \nVdash D \preccurlyeq A$ for every $D \in \Sigma_2$ and the formula interpretation of the second premiss of jump does not hold in w.

jump: Suppose that for a model $\langle W, N, [\![\,]\!]\rangle$ and $w \in W$ we have $w \nVdash \bigwedge \Gamma \to \bigvee \Delta \vee \bigvee_{B \in \Sigma} B \preccurlyeq A$. Then there must be a sphere $\alpha \in N(w)$ such that $[\![A]\!] \cap \alpha \neq \emptyset$ and $[\![B]\!] \cap \alpha = \emptyset$ for every $B \in \Sigma$. Hence there is a world $v \in W$ with $v \Vdash A \wedge \neg \bigvee_{B \in \Sigma} B$, and thus at this world the formula interpretation of the premiss of jump does not hold.

N: Suppose that for a normal model $\langle W, N, [\![\,]\!]\rangle$ and $w \in W$ we have $w \nVdash \bigwedge \Gamma \to \bigvee \Delta$. By normality we have that $N(w) \neq \emptyset$, and hence $w \nVdash \bot \preccurlyeq \top$. Thus the formula interpretation of the premiss of N does not hold at w.

T: Suppose that for a totally reflexive model $\langle W, N, [\![\,]\!]\rangle$ and $w \in W$ we have $w \nVdash \bigwedge\Gamma \wedge A \preccurlyeq B \to \bigvee\Delta$. If $w \nVdash B$, the formula interpretation of the first premiss of T does not hold at w. Otherwise we have $w \Vdash B$ and by total reflexivity thus also $[\![B]\!] \cap \bigcup N(w) \neq \emptyset$. Together with $w \Vdash A \preccurlyeq B$ we thus obtain $[\![A]\!] \cap \bigcup N(w) \neq \emptyset$. But then $w \nVdash \bot \preccurlyeq A$ and the formula interpretation of the second premiss of T does not hold at w.

W: Suppose that for a weakly centered model $\langle W, N, [\![\,]\!]\rangle$ and $w \in W$ we have $w \nVdash \bigwedge\Gamma \to \bigvee\Delta \vee \bigvee_{B\in\Sigma} B \preccurlyeq A$. Since the model is weakly centered, we have $N(w) \neq \emptyset$ and $w \in \bigcap N(w)$. But then for every $B \in \Sigma$ we have $w \nVdash B$, since otherwise $w \Vdash B \preccurlyeq A$. Thus the formula interpretation of the premiss of W does not hold at w.

C: Suppose that for a centered model $\langle W, N, [\![\,]\!]\rangle$ and $w \in W$ we have $w \nVdash \bigwedge\Gamma \wedge C \preccurlyeq D \to \bigvee\Delta$. If $w \nVdash D$, the formula interpretation of the second premiss of C does not hold in w. Otherwise, since the model is centered we have that $\{w\} \in N(w)$. From $\{w\} \cap [\![D]\!] \neq \emptyset$ and $w \Vdash C \preccurlyeq D$, we also have $w \Vdash C$ and the formula interpretation of the first premiss does not hold in w.

A: Suppose that for an absolute model $\langle W, N, [\![\,]\!]\rangle$ and $w \in W$ we have $w \nVdash \bigwedge\Gamma \to \bigvee\Delta \vee \bigvee_{B\in\Sigma} B \preccurlyeq A$. Then as in the case of the jump rule there must be a sphere $\alpha \in N(w)$ with $[\![A]\!] \cap \alpha \neq \emptyset$ and $[\![B]\!] \cap \alpha = \emptyset$ for every $B \in \Sigma$, and hence there is a world $v \in W$ with $v \Vdash A \wedge \neg\bigvee_{B\in\Sigma} B$. Moreover, since the model is absolute, we have $N(v) = N(w)$, and hence for every formula of the form $C \preccurlyeq D$ we have $w \Vdash C \preccurlyeq D$ if and only if $v \Vdash C \preccurlyeq D$. Taking these together we have $v \nVdash \bigwedge\Gamma^{\preccurlyeq}, A \to \bigvee\Delta^{\preccurlyeq}, \Sigma$, and hence the formula interpretation of the premiss of A does not hold at v. □

4.3 Cut elimination for $\mathcal{I}_{\mathbb{V}}$

In this section we shall prove cut elimination for the sequent calculus $\mathcal{I}_{\mathbb{V}}$, the calculus for the basic system, in presence of the contraction rules $\mathsf{Ctr_L}$, $\mathsf{Ctr_R}$, $\mathsf{Ctr_B}$ and $\mathsf{Ctr_i}$. The cut rules are:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{cut}_1 \qquad \frac{\Gamma \Rightarrow \Delta, [\Omega \lhd A] \quad \Sigma \Rightarrow \Pi, [A, \Theta \lhd B]}{\Gamma, \Sigma \Rightarrow \Delta, \Pi\, [\Omega, \Theta \lhd B]}\ \mathsf{cut}_2$$

The general strategy is adapted from the hypersequent setting [21], and it is composed of three lemmas. Lemma cut_1-reduction shows how to eliminate occurrences of cut_1. To eliminate occurrences of cut_2 we need two lemmas. One (cut_2-reduction) permutes the cut upwards on the left premiss until an occurrence of jump is reached. When such an occurrence is reached, lemma shift-right eliminates it, by permuting it upwards on the right premiss.
We write Γ, Δ for multiset union and A^n for the multiset containing n copies of the formula A.

Definition 4.3.1. We write $\mathcal{I}_{\mathsf{Base}}(\mathsf{c})$ for the calculus $\mathcal{I}_{\mathsf{Base}}$ extended with the cut rules cut_1 and cut_2. The *complexity* of an application of cut_1 or cut_2 is the complexity of the cut formula. Given a derivation $\mathcal{D}$ in $\mathcal{I}_{\mathsf{Base}}(\mathsf{c})$, its *formula cut rank* $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D})$ is the maximal complexity of an application of cut_1 in it. Analogously, its *structural cut rank* $\mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D})$ is the maximal complexity of an application of cut_2 in it.

Definition 4.3.2. As in the previous chapters, the *height* of a derivation is the number of nodes of its longest branch minus one. We write $\mathcal{I}_{\mathsf{Base}} \vdash^n \Gamma \Rightarrow \Delta$ if there exists a derivation of height n in $\mathcal{I}_{\mathsf{Base}}$ with endsequent $\Gamma \Rightarrow \Delta$. Similarly for $\mathcal{I}_{\mathsf{Base}}(\mathsf{c})$.

Lemma 4.3.1. The *weakening rules* are height-preserving admissible in $\mathcal{I}_{\mathsf{Base}}$ and $\mathcal{I}_{\mathsf{Base}}(\mathsf{c})$, i.e. (using the uniform notation $\mathcal{I}_{\mathsf{Base}}(\mathsf{c})$ for both cases): If $\mathcal{I}_{\mathsf{Base}}(\mathsf{c}) \vdash^n \Gamma \Rightarrow \Delta$, then $\mathcal{I}_{\mathsf{Base}}(\mathsf{c}) \vdash^n \Gamma, \Sigma \Rightarrow \Delta, \Pi$ and if $\mathcal{I}_{\mathsf{Base}}(\mathsf{c}) \vdash^n \Gamma \Rightarrow \Delta, [\Sigma \lhd A]$, then $\mathcal{I}_{\mathsf{Base}}(\mathsf{c}) \vdash^n \Gamma \Rightarrow \Delta, [\Sigma, \Omega \lhd A]$. Moreover, the formula cut ranks are preserved.

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{Wk} \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}{\Gamma \Rightarrow \Delta, [\Sigma, \Omega \lhd A]}\ \mathsf{Wk_i}$$

Proof. We prove both statements simultaneously by inductive on the height of the derivation, distinguishing cases according to the last applied rule. We sketch some representative cases.

jump: Suppose that $\mathcal{I}_{\mathsf{Base}} \vdash^{n+1} \Gamma \Rightarrow \Delta, [\Sigma \lhd A]$ with last applied rule jump on the displayed block. Then $\mathcal{I}_{\mathsf{Base}} \vdash^n A \Rightarrow \Sigma$. For weakening on the formula level we simply apply the jump rule with the weakened context to obtain $\mathcal{I}_{\mathsf{Base}} \vdash^{n+1} \Gamma, \Omega \Rightarrow \Delta, \Pi, [\Sigma \lhd A]$. For weakening inside blocks we apply the inductive hypothesis on the statement 4.3.1 to obtain $\mathcal{I}_{\mathsf{Base}} \vdash^n A \Rightarrow \Sigma, \Omega$. Now an application of jump gives $\mathcal{I}_{\mathsf{Base}} \vdash^{n+1} \Gamma \Rightarrow \Delta, [\Sigma, \Omega \lhd A]$.

$\mathsf{Ctr_i}$: Suppose that $\mathcal{I}_{\mathsf{Base}} \vdash^{n+1} \Gamma \Rightarrow \Delta, [\Sigma \lhd A]$ with last applied rule $\mathsf{Ctr_i}$. Then $\mathcal{I}_{\mathsf{Base}} \vdash^n \Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd A]$. For weakening on the formula level we apply the inductive hypothesis about statement 4.3.1 to obtain $\mathcal{I}_{\mathsf{Base}} \vdash^n \Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Sigma \lhd A]$, then apply the rule $\mathsf{Ctr_i}$. For weakening inside blocks we apply the inductive hypothesis about statement 4.3.1 twice to obtain $\mathcal{I}_{\mathsf{Base}} \vdash^n \Gamma \Rightarrow \Delta, [\Sigma, \Omega \lhd A], [\Sigma, \Omega \lhd A]$, and again an application of $\mathsf{Ctr_i}$ gives the desired result.

$\mathsf{cut_2}$: Suppose that $\mathcal{I}_{\mathsf{Base}}(\mathsf{c}) \vdash^{n+1} \Gamma, \Sigma \Rightarrow \Delta, \Pi\, [\Omega, \Theta \lhd B]$ with last applied rule $\mathsf{cut_2}$. Then $\mathcal{I}_{\mathsf{Base}}(\mathsf{c}) \vdash^n \Gamma \Rightarrow \Delta, [\Omega \lhd A]$ and $\mathcal{I}_{\mathsf{Base}}(\mathsf{c}) \vdash_n \Sigma \Rightarrow \Pi, [A, \Theta \lhd B]$. For weakening on the formula level we apply the inductive hypothesis on the first premiss to obtain $\mathcal{I}_{\mathsf{Base}}(\mathsf{c}) \vdash^n \Gamma, \Gamma' \Rightarrow \Delta, \Delta', [\Omega \lhd A]$, then apply the rule $\mathsf{cut_2}$. For weakening inside the active block again we apply the inductive hypothesis on the first premiss to obtain $\mathcal{I}_{\mathsf{Base}}(\mathsf{c}) \vdash^n \Gamma \Rightarrow \Delta, [\Omega, \Omega' \lhd A]$, followed by an application of $\mathsf{cut_2}$.

Since none of the transformations involves a cut formula, the cut ranks are preserved. □

Lemma 4.3.2 ($\mathsf{cut_1}$-reduction)**.** Suppose $\mathcal{I}_{\mathbb{V}}(\mathsf{c}) \vdash \Gamma \Rightarrow \Delta, A^n$ and $\mathcal{I}_{\mathbb{V}}(\mathsf{c}) \vdash A^m, \Sigma \Rightarrow \Pi$ by derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ with $\mathsf{rk}_{\mathsf{cut_1}}(\mathcal{D}_1) < |A|$, $\mathsf{rk}_{\mathsf{cut_1}}(\mathcal{D}_2) < |A|$ and $\mathsf{rk}_{\mathsf{cut_2}}(\mathcal{D}_1) < |A|$ $\mathsf{rk}_{\mathsf{cut_2}}(\mathcal{D}_2) < |A|$. A^n and A^m denote n and m occurrences of A. Then there is a derivation $\mathcal{D}$ in $\mathcal{I}_{\mathbb{V}}(\mathsf{c})$ of $\Gamma, \Sigma \Rightarrow \Delta, \Pi$ with $\mathsf{rk}_{\mathsf{cut_1}}(\mathcal{D}) < |A|$ and $\mathsf{rk}_{\mathsf{cut_2}}(\mathcal{D}) < |A|$.

Proof. By inductive on the sum of the heights of the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$. We count the atom p in an application of init, formula $\bot$ in an application of $\bot_\mathsf{L}$ as well as the contracted formula in the contraction rules as principal. We distinguish three main cases: (i) none of the occurrences of A is principal in the last rule application of $\mathcal{D}_1$; (ii) at least one of the occurrences of A is principal in the last rule applied in $\mathcal{D}_1$ and none of the occurrences of A is principal in the last applied rule in $\mathcal{D}_2$; (iii) at least one occurrence of A is principal in the last rule applied both in $\mathcal{D}_1$ and $\mathcal{D}_2$.

In case (i) we apply the inductive hypothesis on the premiss(es) of the last applied rule R in $\mathcal{D}_1$ and the conclusion of $\mathcal{D}_2$, then apply the rule R to obtain the desired conclusion. This is possible since none of the rules poses any restrictions on the context.

In case (ii) we apply the inductive hypothesis on the premiss(es) of the last applied rule R in $\mathcal{D}_2$, followed by an application of the same rule to obtain the desired conclusion. Again, this is possible since none of the rules poses any restrictions on the context.

In case (iii) we distinguish cases according to whether the last rules applied in $\mathcal{D}_1$ and $\mathcal{D}_2$ are instances of contraction or of logical rules. If the last rule in $\mathcal{D}_1$ is $\mathsf{Ctr_R}$ with an occurrence of A principal, we simply apply the inductive hypothesis to its premiss and are done. Similarly if the last rule in $\mathcal{D}_2$ is $\mathsf{Ctr_L}$ with an occurrence of A principal. So suppose that at least one occurrence of A is principal in the last applied rules of both $\mathcal{D}_1$ and $\mathcal{D}_2$, and that neither of the rules is a contraction rule. We distinguish cases according to the form of A.

- $A = p$. In this case the last applied rules in both $\mathcal{D}_1$ and $\mathcal{D}_2$ is init and the conclusion of the cut is derived by init as well.

- $A = B \to C$. The derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ end with:

$$\dfrac{\begin{matrix}\mathcal{D}_1' \\ \vdots \\ \Gamma, B \Rightarrow \Delta, (B \to C)^{n-1}, C\end{matrix}}{\Gamma \Rightarrow \Delta, (B \to C)^{n-1}, B \to C} \to_R \qquad \dfrac{\begin{matrix}\mathcal{D}_2' \\ \vdots \\ C, (B \to C)^{m-1}, \Sigma \Rightarrow \Pi\end{matrix} \quad \begin{matrix}\mathcal{D}_2'' \\ \vdots \\ (B \to C)^{m-1}, \Sigma \Rightarrow B, \Pi\end{matrix}}{B \to C, (B \to C)^{m-1}, \Sigma \Rightarrow \Pi} \to_L$$

We first apply "cross-cuts", i.e., the inductive hypothesis on the premiss of $\to_R$ and the conclusion of $\to_L$ as well as on the conclusion of $\to_R$ and the premisses of $\to_L$ to obtain derivations of

$$\Gamma, B, \Sigma \Rightarrow \Delta, C, \Pi \qquad \Gamma, C, \Sigma \Rightarrow \Delta, \Pi \qquad \Gamma, \Sigma \Rightarrow \Delta, B, \Pi$$

Now two applications of cut_1 yield $\Gamma^3, \Sigma^3 \Rightarrow \Delta^3, \Pi^3$, and with applications of $\mathsf{Ctr_L}$ and $\mathsf{Ctr_R}$ we obtain the desired conclusion. Since $|B| < |B \to C| > |C|$ the restriction on the cut rank is satisfied.

- $A = C \preccurlyeq D$. Then the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ end with:

$$\dfrac{\begin{matrix}\mathcal{D}_1' \\ \vdots \\ \Gamma \Rightarrow \Delta, (C \preccurlyeq D)^{n-1}, [C \lhd D]\end{matrix}}{\Gamma \Rightarrow \Delta, (C \preccurlyeq D)^{n-1}, C \preccurlyeq D} R \preccurlyeq$$

and

$$\dfrac{\begin{matrix}\mathcal{D}_2' \\ \vdots \\ (C \preccurlyeq D)^{m-1}, \Sigma \Rightarrow \Pi, [D, \Omega \lhd B]\end{matrix} \qquad \begin{matrix}\mathcal{D}_2'' \\ \vdots \\ (C \preccurlyeq D)^{m-1}, \Sigma \Rightarrow \Pi, [\Omega \lhd C]\end{matrix}}{C \preccurlyeq D, (C \preccurlyeq D)^{m-1}, \Sigma \Rightarrow \Pi, [\Omega \lhd B]} L \preccurlyeq$$

Again by cross-cuts, i.e., by application of the inductive hypothesis to the premiss of $R \preccurlyeq$ and the conclusion of $L \preccurlyeq$ resp. the conclusion of $R \preccurlyeq$ and the premisses of $L \preccurlyeq$, we obtain derivations of

$$\Gamma, \Sigma \Rightarrow \Delta, \Pi, [C \lhd D] \quad \Gamma, \Sigma \Rightarrow \Delta, \Pi, [D, \Omega \lhd B] \quad \Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega \lhd C]$$

Now applications of cut_2 on C and D give

$$\Gamma^3, \Sigma^2 \Rightarrow \Delta^3, \Pi^3, \left[\Omega^2 \lhd B\right]$$

and we are done using $\mathsf{Ctr_L}, \mathsf{Ctr_R}$ and $\mathsf{Ctr_B}$. Again the newly introduced cuts are on lower complexity. $\square$

Lemma 4.3.3 (Shift-right)**.** Suppose for $k_1, \ldots, k_n \geq 1$ we have $\mathcal{I}_{\mathbb{V}}(\mathsf{c})$-derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\Gamma \Rightarrow \Delta, [\Omega \lhd A]$ and $\Sigma \Rightarrow \Pi, \left[A^{k_1}, \Theta_1 \lhd B_1\right], \ldots, \left[A^{k_n}, \Theta_n \lhd B_n\right]$ such that the last applied rule in $\mathcal{D}_1$ is jump, and with $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}_1) \leq |A|$, $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}_2) \leq |A|$ and $\mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D}_1) < |A|$, $\mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D}_2) < |A|$. Then there is a derivation $\mathcal{D}$ in $\mathcal{I}_{\mathbb{V}}(\mathsf{c})$ with $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}) \leq |A|$ $\mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D}) < |A|$ of the sequent

$$\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega, \Theta_1 \lhd B_1], \ldots, [\Omega, \Theta_n \lhd B_n]$$

Proof. By inductive on the height of $\mathcal{D}_2$, distinguishing cases according to the last applied rule in $\mathcal{D}_2$.

If the last applied rule in $\mathcal{D}_2$ was jump on one of the blocks containing A (w.l.o.g. on the first one), we have the situation

$$\dfrac{\begin{array}{c}\mathcal{D}_1' \\ \vdots \\ A \Rightarrow \Sigma\end{array}}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{jump} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2' \\ \vdots \\ B_1 \Rightarrow A^{k_1}, \Pi_1\end{array}}{\Omega \Rightarrow \Theta, \left[A^{k_1}, \Pi_1 \lhd B_1\right], \ldots, \left[A^{k_n}, \Pi_n \lhd B_n\right]}\ \mathsf{jump}$$

This is converted into the derivation

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} \dfrac{\begin{array}{c}\mathcal{D}_2' \\ \vdots \\ B_1 \Rightarrow A^{k_1}, \Pi_1\end{array} \qquad \begin{array}{c}\mathcal{D}_1' \\ \vdots \\ A \Rightarrow \Sigma\end{array}}{B_1 \Rightarrow \Sigma, A^{k_1 - 1}, \Pi_1}\ \mathsf{cut}_1 \\ \vdots \\ B_1 \Rightarrow \Sigma^{k_1 - 1}, A, \Pi_1 \end{array} \qquad \begin{array}{c}\mathcal{D}_1' \\ \vdots \\ A \Rightarrow \Sigma\end{array}}{B_1 \Rightarrow \Sigma^{k_1}, \Pi_1}\ \mathsf{cut}_1}{B_1 \Rightarrow \Sigma, \Pi_1}\ \mathsf{Ctr_R}}{\Gamma, \Omega \Rightarrow \Delta, \Theta, [\Sigma, \Pi_1 \lhd B_1], \ldots, [\Sigma, \Pi_n \lhd B_n]}\ \mathsf{jump}$$

Where contraction takes care of all k_1 occurrences of Σ which have been introduced in the process, and all applications of cut_1 have complexity $|A|$.

If the last applied rule in $\mathcal{D}_2$ is a propositional rule, $R \preccurlyeq$, or one of the cut rules we apply the inductive hypothesis (on the height of $\mathcal{D}_2$) to the premiss and then apply the same rule, since none of these rules contains an active block in the conclusion. E.g., if the last applied rule in $\mathcal{D}_2$ was $\to_R$, we have the situation

$$\dfrac{A \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{jump} \qquad \dfrac{\Omega, C \Rightarrow \Theta, D, [A, \Pi_1 \lhd B_1], \ldots, [A, \Pi_n \lhd B_n]}{\Omega \Rightarrow \Theta, C \to D, \left[A^{k_1}, \Pi_1 \lhd B_1\right], \ldots, \left[A^{k_n}, \Pi_n \lhd B_n\right]}\ \to_R$$

Using the inductive hypothesis and applying the rule $\to_R$ we then have a derivation ending in

$$\dfrac{\Gamma, \Omega, C \Rightarrow \Delta, \Theta, D, [\Sigma, \Pi_1 \lhd B_1], \ldots, [\Sigma, \Pi_n \lhd B_n]}{\Gamma, \Omega \Rightarrow \Delta, \Theta, C \to D, [\Sigma, \Pi_1 \lhd B_1], \ldots, [\Sigma, \Pi_n \lhd B_n]}\ \to_R$$

The situation is more interesting if the last applied rule in $\mathcal{D}_2$ was com or $L \preccurlyeq$. In the latter case the derivation $\mathcal{D}_2$ ends in

$$\dfrac{\begin{array}{l}\Omega \Rightarrow \Theta, \left[A^{k_1}, \Pi_1 \lhd C\right], \left[A^{k_2}, \Pi_2 \lhd B_2\right], \ldots, \left[A^{k_n}, \Pi_n \lhd B_n\right] \\ \qquad\qquad \Omega \Rightarrow \Theta, \left[A^{k_1}, \Pi_1, D \lhd B_1\right], \left[A^{k_2}, \Pi_2 \lhd B_2\right], \ldots, \left[A^{k_n}, \Pi_n \lhd B_n\right]\end{array}}{\Omega, C \preccurlyeq D \Rightarrow \Theta, \left[A^{k_1}, \Pi_1 \lhd B_1\right], \ldots, \left[A^{k_n}, \Pi_n \lhd B_n\right]}\ L \preccurlyeq$$

Now applying a cross cut, i.e., the inductive hypothesis on the conclusion of jump and the two premisses of this rule application and then applying $L \preccurlyeq$ we obtain a derivation ending in

$$\dfrac{\Gamma, \Omega \Rightarrow \Delta, \Theta, [\Sigma, \Pi_1 \lhd C], [\Sigma, \Pi_2 \lhd B_2], \ldots, [\Sigma, \Pi_n \lhd B_n] \qquad \Gamma, \Omega \Rightarrow \Delta, \Theta, [\Sigma, \Pi_1, D \lhd B_1], [\Sigma, \Pi_2 \lhd B_2], \ldots, [\Sigma, \Pi_n \lhd B_n]}{\Gamma, \Omega, C \preccurlyeq D \Rightarrow \Delta, \Theta, [\Sigma, \Pi_1 \lhd B_1], \ldots, [\Sigma, \Pi_n \lhd B_n]}\ L \preccurlyeq$$

The inductive hypothesis ensures that the resulting derivation satisfies the restrictions on the cut ranks.

The case of the last rule in $\mathcal{D}_2$ being com is similar: assuming the cut formula A occurs in both active blocks, the derivation $\mathcal{D}_2$ ends in

$$\dfrac{\Omega \Rightarrow \Theta, \left[A^{k_1+k_2}, \Pi_1, \Pi_2 \lhd B_1\right], \left[A^{k_3}, \Pi_3 \lhd B_3\right], \ldots, \left[A^{k_n}, \Pi_n \lhd B_n\right] \qquad \Omega \Rightarrow \left[A^{k_1+k_2}, \Pi_1, \Pi_2 \lhd B_2\right], \left[A^{k_3}, \Pi_3 \lhd B_3\right], \ldots, \left[A^{k_n}, \Pi_n \lhd B_n\right]}{\Omega \Rightarrow \Theta, \left[A^{k_1}, \Pi_1 \lhd B_1\right], \left[A^{k_2}, \Pi_2 \lhd B_2\right], \left[A^{k_3}, \Pi_3 \lhd B_3\right], \ldots, \left[A^{k_n}, \Pi_n \lhd B_n\right]}\ \mathsf{com}$$

Using the inductive hypothesis we hence obtain derivations of

$$\Gamma, \Omega \Rightarrow \Delta, \Theta, [\Sigma, \Pi_1, \Pi_2 \lhd B_1], [\Sigma, \Pi_3 \lhd B_3], \ldots, [\Sigma, \Pi_n \lhd B_n]$$

and

$$\Gamma, \Omega \Rightarrow \Delta, \Theta, [\Sigma, \Pi_1, \Pi_2 \lhd B_2], [\Sigma, \Pi_3 \lhd B_3], \ldots, [\Sigma, \Pi_n \lhd B_n].$$

now admissibility of weakening inside blocks followed by an application of com yields the desired conclusion. Again the inductive hypothesis ensures the restrictions on the cut formulas.

If the last applied rule in $\mathcal{D}_2$ was $\mathsf{Ctr_B}$ or $\mathsf{Ctr_i}$, we simply apply the inductive hypothesis on the premiss of this rule. □

Lemma 4.3.4 (cut_2-reduction)**.** Suppose we have $\mathcal{I}_{\mathbb{V}}$-derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\Gamma \Rightarrow \Delta, [\Omega_1 \lhd A], \ldots, [\Omega_n \lhd A]$ and $\Sigma \Rightarrow \Pi, [A, \Theta \lhd B]$ with $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}_1) \leq |A|$, $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}_2) \leq |A|$ and $\mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D}_1) < |A|$, $\mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D}_2) < |A|$. Then there is a derivation $\mathcal{D}$ in $\mathcal{I}_{\mathbb{V}}(\mathsf{c})$ with $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}) \leq |A|$, $\mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D}) < |A|$ of the sequent

$$\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega_1, \Theta \lhd B], \ldots, [\Omega_n, \Theta \lhd B]$$

Proof. By inductive on the height of $\mathcal{D}_1$, distinguishing cases according to the last applied rule in $\mathcal{D}_1$.

If the last applied rule in $\mathcal{D}_1$ was jump, then we apply Lem. 4.3.3 and are done.

If the last applied rule in $\mathcal{D}_1$ is such that none of the occurrences of A is in an active block in its conclusion, then we simply apply the inductive hypothesis on its premiss(es), followed by an application of the same rule. E.g., if the last applied rule was $R \preccurlyeq$, the derivation $\mathcal{D}_1$ ends in

$$\dfrac{\begin{array}{c}\mathcal{D}_1' \\ \vdots \\ \Gamma \Rightarrow \Delta, [C \lhd D], [\Omega_1 \lhd A], \ldots, [\Omega_n \lhd A]\end{array}}{\Gamma \Rightarrow \Delta, C \preccurlyeq D, [\Omega_1 \lhd A], \ldots, [\Omega_n \lhd A]}\ R \preccurlyeq$$

We apply the inductive hypothesis to the premiss of $R \preccurlyeq$ to obtain

$$\Gamma, \Sigma \Rightarrow \Delta, \Pi, [C \lhd D], [\Omega_1, \Theta \lhd B], \ldots, [\Omega_n, \Theta \lhd B]$$

and an application of $R \preccurlyeq$ yields the desired result. Note that this case covers in particular the propositional rules as well as the cut rules and contractions on formulas.

If the last applied rule was com where both active components contain an occurrence of A, the derivation $\mathcal{D}_1$ ends in

$$\dfrac{\begin{array}{c}\Gamma \Rightarrow \Delta, [\Omega_1 \lhd A], \ldots, [\Omega_{n-2} \lhd A], [\Omega_{n-1}, \Omega_n \lhd A] \\ \Gamma \Rightarrow \Delta, [\Omega_1 \lhd A], \ldots, [\Omega_{n-2} \lhd A], [\Omega_{n-1}, \Omega_n \lhd A]\end{array}}{\Gamma \Rightarrow \Delta, [\Omega_1 \lhd A], \ldots, [\Omega_{n-2} \lhd A], [\Omega_{n-1} \lhd A], [\Omega_n \lhd A]} \ \mathsf{com}$$

Applying the inductive hypothesis to the premisses of this rule yields

$$\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega_1, \Theta \lhd B], \ldots, [\Omega_{n-2}, \Theta \lhd B], [\Omega_{n-1}, \Omega_n, \Theta \lhd B]$$

Now admissibility of weakening yields

$$\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega_1, \Theta \lhd B], \ldots, [\Omega_{n-2}, \Theta \lhd B], [\Omega_{n-1}, \Theta, \Omega_n, \Theta \lhd B]$$

and an application of com gives the desired result.

If the last applied rule was $L \preccurlyeq$ with the occurrence of A in the active block, the derivation $\mathcal{D}_1$ ends in

$$\dfrac{\begin{array}{c}\Gamma \Rightarrow \Delta, [\Omega_1 \lhd A], \ldots, [\Omega_{n-1} \lhd A], [D, \Omega_n \lhd A] \\ \Gamma \Rightarrow \Delta, [\Omega_1 \lhd A], \ldots, [\Omega_{n-1} \lhd A], [\Omega_n \lhd C]\end{array}}{\Gamma, C \preccurlyeq D \Rightarrow \Delta, [\Omega_1 \lhd A], \ldots, [\Omega_{n-1} \lhd A], [\Omega_n \lhd A]} \ L \preccurlyeq$$

Cross-cuts give the derivations of

$$\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega_1, \Theta \lhd B], \ldots, [\Omega_{n-1}, \Theta \lhd B], [D, \Omega_n, \Theta \lhd B]$$

and

$$\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega_1, \Theta \lhd B], \ldots, [\Omega_{n-1}, \Theta \lhd B], [\Omega_n \lhd C]$$

Now admissibility of weakening on the second premiss yields

$$\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega_1, \Theta \lhd B], \ldots, [\Omega_{n-1}, \Theta \lhd B], [\Omega_n, \Theta \lhd C]$$

and an application of $L \preccurlyeq$ gives the result.

Finally, if the last applied rule was $\mathsf{Ctr_B}$, we simply apply the inductive hypothesis to its premiss. □

Theorem 4.3.5 (Cut elimination)**.** If $\mathcal{I}_{\mathbb{V}}(\mathsf{c}) \vdash \Gamma \Rightarrow \Delta$, then $\mathcal{I}_{\mathbb{V}} \vdash \Gamma \Rightarrow \Delta$. In particular, there is a procedure to eliminate cuts from a derivation in $\mathcal{I}_{\mathbb{V}}(\mathsf{c})$.

Proof. Let $\mathcal{D}$ be a derivation in $\mathcal{I}_{\mathbb{V}}\mathsf{c}$. Let $\#_{\mathsf{cut}_1}(\mathcal{D})$ be the number of applications of cut_1 in $\mathcal{D}$ with cut formula of complexity $\max\{\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}), \mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D})\}$, and analogous for $\#_{\mathsf{cut}_2}(\mathcal{D})$ with respect to applications of cut_2. We show how to obtain a derivation $\mathcal{D}'$ in $\mathcal{I}_{\mathbb{V}}$ with the same endsequent by inductive on the tuples $\langle \mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}), \#_{\mathsf{cut}_2}(\mathcal{D}), \#_{\mathsf{cut}_1}(\mathcal{D})\rangle$ in the lexicographic ordering.

A topmost application of cut_1 with complexity $\max\{\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}), \mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D})\}$ is eliminated using the cut_1-reduction lemma (lemma 4.3.2), either keeping $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D})$ and $\#_{\mathsf{cut}_2}(\mathcal{D})$ the same and decreasing $\#_{\mathsf{cut}_1}(\mathcal{D})$ or decreasing $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D})$ while possibly increasing $\#_{\mathsf{cut}_2}(\mathcal{D})$ and $\#_{\mathsf{cut}_1}(\mathcal{D})$. A topmost application of cut_2 with complexity $\max\{\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}), \mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D})\}$ is eliminated using the cut_2-reduction lemma (lemma 4.3.4), keeping $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D})$ the same and decreasing $\#_{\mathsf{cut}_2}(\mathcal{D})$, while possibly increasing $\#_{\mathsf{cut}_1}(\mathcal{D})$. □

As a consequence of cut-elimination, we can syntactically prove completeness of logic $\mathbb{V}$.

Theorem 4.3.6. If a formula F is valid in $\mathbb{V}$, then there is a derivation of $\Rightarrow F$ in $\mathcal{I}_{\mathbb{V}}$.

Proof. By deriving the axioms of the Hilbert-calculus for $\mathbb{V}$ (Figure 4.1) in $\mathcal{I}_{\mathbb{V}}(\mathsf{c})$ and showing that the calculus is closed with respect to the inference rules, namely that if the premisses of an inference rule are derivable, then so is the conclusion. The proof makes use of the cut-elimination theorem (Theorem 4.3.5).

As for rule (CPR), we have to show that if the premiss $B \to A$ of (CPR) is derivable, then so is the conclusion $A \preccurlyeq B$. Assume $\Rightarrow A \to B$ is derivable. By admissibility of weakening, sequent $B, B \to A \Rightarrow A$ is derivable:

$$\dfrac{B \Rightarrow A, B \quad B, A \Rightarrow A}{B, B \to A \Rightarrow A}\ \to_L$$

Then again, by derivability of $B \to A$ and admissibility of weakening, we have that $B \Rightarrow A, B \to A$ is derivable, from which we have:

$$\dfrac{\dfrac{\dfrac{B \Rightarrow A, B \to A \quad B, B \to A \Rightarrow A}{B \Rightarrow A}\ \mathsf{cut}_1}{\Rightarrow [A \lhd B]}\ \mathsf{jump}}{\Rightarrow A \preccurlyeq B}\ \mathsf{R}\preccurlyeq^{\mathsf{i}}$$

The derivation of *modus ponens* employs the rules of propositional logic and cut_1 (standard). The axioms of $\mathbb{V}$ are derived in the sequent calculus as follows.

(CPA)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \overset{\triangledown}{\Rightarrow} A, B \quad B \overset{\triangledown}{\Rightarrow} A, B}{A \vee B \Rightarrow A, B}\ \vee_L}{\Rightarrow [A, B \lhd A \vee B], [B \lhd A \vee B]}\ \mathsf{jump} \quad \dfrac{\dfrac{A \overset{\triangledown}{\Rightarrow} A, B \quad B \overset{\triangledown}{\Rightarrow} A, B}{A \vee B \Rightarrow A, B}\ \vee_L}{\Rightarrow [A \lhd A \vee B], [A, B \lhd A \vee B]}\ \mathsf{jump}}{\Rightarrow [A \lhd A \vee B], [B \lhd A \vee B]}\ \mathsf{com}^{\mathsf{i}}}{\Rightarrow [A \lhd A \vee B], B \preccurlyeq A \vee B}\ \mathsf{R}\preccurlyeq^{\mathsf{i}}}{\Rightarrow A \preccurlyeq A \vee B, B \preccurlyeq A \vee B}\ \mathsf{R}\preccurlyeq^{\mathsf{i}}}{\Rightarrow (A \preccurlyeq A \vee B) \vee (B \preccurlyeq A \vee B)}\ \vee_R$$

(TR)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \overset{\triangledown}{\Rightarrow} A}{\cdots \Rightarrow [A \lhd C], [A \lhd A]}\ \mathsf{jump} \quad \dfrac{\dfrac{B \overset{\triangledown}{\Rightarrow} B, A}{\cdots \Rightarrow [B, A \lhd C], [B, A \lhd B]}\ \mathsf{jump} \quad \dfrac{C \overset{\triangledown}{\Rightarrow} C, B, A}{\cdots \Rightarrow [C, B, A \lhd C]}\ \mathsf{jump}}{A \preccurlyeq B, B \preccurlyeq C \Rightarrow [B, A \lhd C]}\ \mathsf{L}\preccurlyeq^{\mathsf{i}}}{A \preccurlyeq B, B \preccurlyeq C \Rightarrow [A \lhd C]}\ \mathsf{L}\preccurlyeq^{\mathsf{i}}}{A \preccurlyeq B, B \preccurlyeq C \Rightarrow A \preccurlyeq C}\ \mathsf{R}\preccurlyeq^{\mathsf{i}}}{\dfrac{(A \preccurlyeq B) \wedge (B \preccurlyeq C) \Rightarrow A \preccurlyeq C}{\Rightarrow (A \preccurlyeq B) \wedge (B \preccurlyeq C) \to (A \preccurlyeq C)}\ \to_R}\ \wedge_L$$

(CO)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \overset{\triangledown}{\Rightarrow} A, B}{\Rightarrow [A, B \lhd B], [B \lhd A]}\ \mathsf{jump} \quad \dfrac{A \overset{\triangledown}{\Rightarrow} A, B}{\Rightarrow [A \lhd B], [A, B \lhd A]}\ \mathsf{jump}}{\Rightarrow [A \lhd B], [B \lhd A]}\ \mathsf{com}^{\mathsf{i}}}{\Rightarrow [A \lhd B], B \preccurlyeq A}\ \mathsf{R}\preccurlyeq^{\mathsf{i}}}{\Rightarrow A \preccurlyeq B, B \preccurlyeq A}\ \mathsf{R}\preccurlyeq^{\mathsf{i}}}{\Rightarrow (A \preccurlyeq B) \vee (B \preccurlyeq A)}\ \vee_R$$

□

4.4 An invertible version: $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$

In this section we present an invertible version of the proof systems for Base presented in the previous section. We call $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$ the invertible calculi. In order to obtain invertibility, the principal formula is repeated in the premiss(es) of the rule. Rules of $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$ can be found in Figure 4.4. The equivalence between $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ is proved via admissibility of weakening and contraction. The invertible versions of the calculi are used to prove semantic completeness.

Initial sequents

$$\frac{}{\Gamma, \bot \Rightarrow \Delta}\ \bot_{\mathsf{L}} \qquad \frac{}{\Gamma, p \Rightarrow \Delta, p}\ \mathsf{init}$$

Propositional and conditional rules

$$\frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}\ \mathsf{L}\to \qquad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ \mathsf{R}\to$$

$$\frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B}\ \mathsf{R}\preccurlyeq^{\mathsf{i}} \qquad \frac{A \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{jump}$$

$$\frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd C]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C]}\ \mathsf{L}\preccurlyeq^{\mathsf{i}}$$

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}\ \mathsf{com}^{\mathsf{i}}$$

Rules for extensions

$$\frac{\Gamma \Rightarrow \Delta, [\bot \lhd \top]}{\Gamma \Rightarrow \Delta}\ \mathsf{N}^{\mathsf{i}} \qquad \frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, B \quad \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\bot \lhd A]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta}\ \mathsf{T}^{\mathsf{i}}$$

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma \lhd A], \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{W}^{\mathsf{i}} \qquad \frac{\Gamma, A \preccurlyeq B, A \Rightarrow \Delta \quad \Gamma, A \preccurlyeq B \Rightarrow B, \Delta}{\Gamma, A \preccurlyeq B \Rightarrow \Delta}\ \mathsf{C}^{\mathsf{i}}$$

$$\frac{\Gamma^{\preccurlyeq}, B \Rightarrow \Delta^{\preccurlyeq}, [\Sigma \lhd B], \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd B]}\ \mathsf{A}^{\mathsf{i}}$$

$\Gamma^{\preccurlyeq} \Rightarrow \Delta^{\preccurlyeq}$ is $\Gamma \Rightarrow \Delta$ restricted to formulas of the form $C \preccurlyeq D$ and blocks.

$$\mathcal{I}^{\mathsf{i}}_{\mathbb{V}} := \{\bot, \mathsf{init}, \mathsf{L}\to, \mathsf{R}\to, \mathsf{R}\preccurlyeq^{\mathsf{i}}, \mathsf{L}\preccurlyeq^{\mathsf{i}}, \mathsf{com}^{\mathsf{i}}, \mathsf{jump}\}$$

$$\mathcal{I}^{\mathsf{i}}_{\mathbb{VN}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}^{\mathsf{i}}\} \qquad \mathcal{I}^{\mathsf{i}}_{\mathbb{VW}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}^{\mathsf{i}}, \mathsf{T}^{\mathsf{i}}, \mathsf{W}^{\mathsf{i}}\} \qquad \mathcal{I}^{\mathsf{i}}_{\mathbb{VA}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{A}^{\mathsf{i}}\}$$
$$\mathcal{I}^{\mathsf{i}}_{\mathbb{VT}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}^{\mathsf{i}}, \mathsf{T}^{\mathsf{i}}\} \qquad \mathcal{I}^{\mathsf{i}}_{\mathbb{VC}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}^{\mathsf{i}}, \mathsf{T}^{i}\mathsf{W}^{\mathsf{i}}, \mathsf{C}^{\mathsf{i}}\} \qquad \mathcal{I}^{\mathsf{i}}_{\mathbb{VNA}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}^{\mathsf{i}}, \mathsf{A}^{\mathsf{i}}\}$$

Figure 4.4: The calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ and its extensions

Lemma 4.4.1. The rules of weakening Wk and $\mathsf{Wk_B}$ are height-preserving admissible in $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$. Also rule $\mathsf{Wk_B}$ (weakening on blocks) is height-preserving admissible.

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{Wk} \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}{\Gamma \Rightarrow \Delta, [\Sigma, \Omega \lhd A]}\ \mathsf{Wk_i} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, [\Sigma \lhd C]}\ \mathsf{Wk_B}$$

Proof. Standard, by simultaneous inductive on the height of the derivation, and by distinguishing cases according to the last rule applied to derive the premiss of the various rules of weakening. $\square$

Lemma 4.4.2. All rules of $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$ are height-preserving invertible, except for the rules of jump and A^{i}.

Proof. We proceed by inductive on the height of the derivation, showing that if the conclusion of a rule (R) is derivable with derivation height h, also its premiss(es) are derivable, with no greater derivation height. Propositional cases and $\mathsf{R} \preccurlyeq^{\mathsf{i}}$ are immediately obtained applying the inductive hypothesis. Invertibility of all the other rules follows by height preserving admissibility of the rules of weakening. Rule jump and A^{i} are not invertible, since some information is lost when going from the conclusion to the premiss of these rules. By means of example, let us consider the rule of jump. Suppose its conclusion $\Gamma \Rightarrow \Delta, [\Sigma \lhd C]$ is derivable. Let $A \preccurlyeq B \in \Gamma$, and suppose that the sequent has been derived by $\mathsf{L} \preccurlyeq^{\mathsf{i}}$.

$$\frac{\Gamma \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma \Rightarrow \Delta, [\Sigma \lhd C], [\Sigma \lhd A]}{\Gamma \Rightarrow \Delta, [\Sigma \lhd C]} \ \mathsf{L} \preccurlyeq^{\mathsf{i}}$$

Applying the inductive hypothesis to the leftmost premiss we have that sequent $C \Rightarrow \Sigma, B$ is derivable, while from the right-hand premiss we non-deterministically obtain that either $A \Rightarrow \Sigma$ is derivable, or that $C \Rightarrow \Sigma$ is derivable. □

Lemma 4.4.3. The four rules of contraction $\mathsf{Ctr}_{\mathsf{L}}$, $\mathsf{Ctr}_{\mathsf{R}}$, $\mathsf{Ctr}_{\mathsf{B}}$ and $\mathsf{Ctr}_{\mathsf{i}}$ are admissible in $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$.

Proof. By mutual inductive on the height of the derivation. All cases are immediate, either by inductive hypothesis or by invertibility of the rule applied to derive the premiss of the contraction rule. To prove admissibility of $\mathsf{Ctr}_{\mathsf{B}}$ we need admissibility of $\mathsf{Ctr}_{\mathsf{L}}$. □

Theorem 4.4.4. For $A \in \mathcal{F}^{\preccurlyeq}$ arbitrary formula, A is derivable in the calculus $\mathcal{I}_{\mathsf{Base}}$ if and only if A is derivable in the invertible calculus $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$.

Proof. Both directions are proved by induction on the height of the derivation, modulo weakening and contraction. For the [if] direction application of weakening is justified, since the rule is admissible in the calculus $\mathcal{I}_{\mathsf{Base}}$, and for direction [$\mathsf{onlyif}$] applications of weakening and contraction are legitimate since the rules are admissible in $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$. □

Standard reasoning shows that the calculi $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$ can be used in a decision procedure for the logic Base as follows. Since contractions and weakenings are admissible we may assume that a derivation of a duplication-free sequent (containing duplicates neither of formulas nor of blocks) only contains duplication-free sequents: whenever a (backwards) application of a rule introduces a duplicate of a formula already in the sequent, it is immediately deleted in the upper sequent by means of weakening. Since all rules have the subformula property, the number of duplication-free sequents possibly relevant to a derivation of a sequent is bounded in the number of subformulas of that sequent, and hence enumerating all possible loop-free derivations of the above form yields a decision procedure for the logic. This argument is sufficient to show termination; however, it is clear that the complexity of the resulting procedure is far from the optimal PSPACE or coNP complexities of the logics [28, 58]. In [80], Olivetti and Pozzato defined a more refined decision procedure for a nested calculus for $\mathbb{V}$ which allows multiple formulas to occur in the consequent of conditional blocks. Their decision procedure is optimal with respect to the PSPACE complexity of $\mathbb{V}$. Thus, we expect also sequent calculi $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ could provide an optimal decision procedure for the logic.

Theorem 4.4.5 (Termination)**.** Proof search for a sequent $\Gamma \Rightarrow \Delta$ in calculus $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$ always comes to an end in a finite number of steps.

Remark 4.4.1. In [36] an automated theorem prover is implemented for logics Base. The prover is called VINTE, for $\mathbb{V}$: INTernal calculi and Extensions, and it is a Prolog implementation of the sequent calculi $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ presented in this section[5]. Given a formula of the language, VINTE is able to decide whether the formula A is valid or not in the chosen logic and, if it is valid, shows a derivation for $\Rightarrow A$. The program is composed of a set of clauses, each implementing a rule of the sequent calculus. At each step of inference, the program checks whether the sequent is an axiom and, if not, the first applicable rule is chosen. The proof search terminates when either there are no more applicable rule, or an instance of an axiom is reached. Refer to [36] for a full description of the program and its functionalities.

4.5 Semantic completeness

In this section we prove the semantic completeness of $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VN}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{T}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VW}}$ and $\mathcal{I}^{\mathsf{i}}_{\mathbb{VC}}$. We do not prove completeness of the calculi with absoluteness, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VA}}$ and $\mathcal{I}^{\mathsf{i}}_{\mathbb{VNA}}$; refer to the end of the section for some remarks on these systems.

In order to simplify the proof we adopt a cumulative version of rules $\rightarrow_L$, $\rightarrow_R$, $\mathsf{R} \preccurlyeq^{\mathsf{i}}$ and $\mathsf{com}^{\mathsf{i}}$. This allows us to consider only the upper sequent of each derivation branch. The proof is by inductive on the modal degree of sequents.

$$\frac{\Gamma, A \rightarrow B, B \Rightarrow \Delta \quad \Gamma, A \rightarrow B \Rightarrow \Delta, B}{\Gamma, A \rightarrow B \Rightarrow \Delta}\ \mathsf{L} \rightarrow_{\mathsf{c}}$$

$$\frac{\Gamma, A \Rightarrow \Delta, A \rightarrow B, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ \mathsf{R} \rightarrow_{\mathsf{c}} \qquad \frac{\Gamma \Rightarrow \Delta, A \preccurlyeq B, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B}\ \mathsf{R} \preccurlyeq^{\mathsf{i}}{}_{\mathsf{c}}$$

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_1 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd B]\, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}\ \mathsf{com}^{\mathsf{i}}_{\mathsf{c}}$$

Definition 4.5.1. The modal degree of a formula resp. sequent is defined as follows:

- $md(\bot) = md(p) = 0$, for p atomic formula;
- $md(A \rightarrow B) = max(md(A), md(B))$;
- $md(A \preccurlyeq B) = max(md(A), md(B)) + 1$;
- $md([\Sigma \lhd A]) = max(md(\Sigma), md(A)) + 1$;
- $md(\Gamma \Rightarrow \Delta) = max\{md(G) \mid G \in \Gamma \cup \Delta, \text{for G formula or block}\}$.

Proposition 4.5.1. All rules of $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$ preserve the modal degree of sequents: the premisses of the rule have a modal degree no greater than the one of the respective conclusion.

Proof. Straightforward, by inspection of the rules of the calculi. Moreover, observe that jump is the only rule which decreases the modal degree. □

An application of a rule is said to be *redundant* if the conclusion of the rule can be derived from one of its premisses by weakening or contraction. If a sequent is derivable it has a non- redundant derivation, since the redundant applications of the rules can be removed without affecting the correctness of the derivation. Moreover, if an application of $\mathsf{com}^{\mathsf{i}}_{\mathsf{c}}$ is non-redundant, then it must respect the following restriction:

$$(*)\ \Sigma_1 \not\subseteq \Sigma_2 \text{ and } \Sigma_2 \not\subseteq \Sigma_1.$$

In fact, if $(*)$ is not respected then either $\Sigma_1 \subseteq \Sigma_2$ or $\Sigma_2 \subseteq \Sigma_1$; in both cases we get a redundant application of $\mathsf{com}^{\mathsf{i}}_{\mathsf{c}}$. All the blocks $[\Sigma_1 \lhd C_1], \ldots, [\Sigma_n \lhd C_n]$ of a saturated

[5] The theorem prover can be found online at `http://193.51.60.97:8000/vinte/`.

sequent can be considered as ordered with respect to set inclusion: once all non redundant $\mathsf{com}^{\mathsf{i}}_{\mathsf{c}}$ have been applied, it holds that either $\Sigma_i \subseteq \Sigma_j$ or $\Sigma_j \subseteq \Sigma_i$. Thus, we order the blocks as follows: $\Sigma_1 \subseteq \Sigma_2 \subseteq ... \subseteq \Sigma_n$.

Definition 4.5.2. Let $\Pi_1 \Rightarrow \Pi_2, [\Sigma_1 \lhd C_1], ..., [\Sigma_n \lhd C_n]$ be a sequent of $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$, with Π_1, Π_2 multi-sets of formulas. The sequent is saturated if it is not an instance of init or $\bot_{\mathsf{L}}$, and it satisfies the following conditions:

1. ($\mathsf{L} \to_{\mathsf{c}}$) if $A \to B \in \Pi_1$ then either $A \in \Pi_2$ or $B \in \Pi_1$;
2. ($\mathsf{R} \to_{\mathsf{c}}$) if $A \to B \in \Pi_2$ then $A \in \Pi_1$ and $B \in \Pi_2$;
3. ($\mathsf{com}^{\mathsf{i}}_{\mathsf{c}}$) for every $[\Sigma_i \lhd C_i]$, $[\Sigma_j \lhd C_j]$ it holds that either $\Sigma_i \subseteq \Sigma_j$ or $\Sigma_j \subseteq \Sigma_i$;
4. ($\mathsf{R} \preccurlyeq^{\mathsf{i}}{}_{\mathsf{c}}$) for every $A \preccurlyeq B \in \Pi_2$ it holds that $[A \lhd B] \in \{[\Sigma_1 \lhd C_1], ..., [\Sigma_n \lhd C_n]\}$;
5. ($\mathsf{L} \preccurlyeq^{\mathsf{i}}$) for every $A \preccurlyeq B \in \Pi_1$ and for every $[\Sigma_i \lhd C_i]$, where $1 \leqslant i \leqslant n$, it holds that either $B \in \Sigma_i$ or there exists $[\Pi, \Sigma \lhd A] \in \{[\Sigma_1 \lhd C_1], ..., [\Sigma_n \lhd C_n]\}$;
6. (N^{i}) either $\Gamma \Rightarrow \Delta$ has the form $\bot \Rightarrow \top$ or $[\bot \lhd \top]$ belongs to Δ;
7. (T^{i}) for all $A \preccurlyeq B$ in Π_1, either $B \in \Pi_2$ or $[\bot \lhd A] \in \{[\Sigma_1 \lhd C_1], ..., [\Sigma_n \lhd C_n]\}$;
8. (W^{i}) for every block $[\Sigma \lhd A]$, it holds that $\Sigma \subseteq \Pi_2$; (C^{i}) for every $A \preccurlyeq B$ in Π_1, it holds that either $B \in \Pi_2$ or $A \in \Pi_1$.

For each logic Base, the definition of saturated sequent takes into account only the saturation conditions of the rules of the corresponding calculus.

We call *static* all the rules except for jump and A^{i}. By *finished* sequent we mean a sequent for which every further static rule application is redundant. Note that a finished sequent is saturated.

Proposition 4.5.2. After finitely many non redundant static rule applications we reach an axiom or a finished sequent.

Proof. Let $\Gamma \Rightarrow \Delta$ be the root sequent of a derivation. We consider any branch of a derivation without applications of jump, and without redundant applications of rules. Observe that each rule application *must* add at least one formula or block to each premiss, and the number of formulas or blocks (each one is finite in itself) that can occur within a sequent is finite. Thus the branch must be finite: if not, then it would not contain axioms and some formula or block would be added infinitely many times by eventually redundant applications of a rule. Moreover, once a rule (R) has been applied to a formula or block, the saturation condition with respect to the rule (R) and the involved formulas or blocks will be satisfied by the premisses of (R). Thus the last node of the branch, if it is not an axiom, must be finished. ☐

Corollary 4.5.3. Given a sequent $\Gamma \Rightarrow \Delta$, every branch of any derivation tree starting with $\Gamma \Rightarrow \Delta$ ends in a finite number of steps with a saturated sequent of no greater modal degree than that of $\Gamma \Rightarrow \Delta$.

Theorem 4.5.4 (Completeness). If a sequent $\Gamma_0 \Rightarrow \Delta_0$ is valid in $\mathbb{V}$, $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$ or $\mathbb{VC}$ then it is derivable in $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VN}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VT}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VW}}$ or $\mathcal{I}^{\mathsf{i}}_{\mathbb{VC}}$.

Proof. We first prove completeness for $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$, then show how to extend the proof to $\mathcal{I}^{\mathsf{i}}_{\mathbb{VN}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VT}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VW}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VC}}$. The proof proceeds by inductive on the modal degree of the sequent, and uses in an essential way the fact that a backwards application of jump reduces the modal degree of a sequent. If $md(\Gamma_0 \Rightarrow \Delta_0) = 0$, $\Gamma_0 \Rightarrow \Delta_0$ is composed only of propositional formulas, and its completeness can be proved from the completeness of sequent calculus for propositional logic. If $md(\Gamma_0 \Rightarrow \Delta_0) > 0$, by Proposition 4.5.1 and Proposition 4.5.3 we have that $\Gamma_0 \Rightarrow \Delta_0$ can be derived from a set of saturated sequents $\Gamma_k \Rightarrow \Delta_k$ of no greater modal degree. Since all the rules are invertible, except jump, and since by hypothesis $\Gamma_0 \Rightarrow \Delta_0$ is valid, also all saturated sequents $\Gamma_k \Rightarrow \Delta_k$ are valid. Thus, either $i)$ $\Gamma_k \Rightarrow \Delta_k$ is an axiom, or $ii)$ it must have been obtained by jump from a valid sequent $\Gamma_{k+1} \Rightarrow \Delta_{k+1}$. In the first case the theorem is trivially proved. We prove $ii)$: if $\Gamma_k \Rightarrow \Delta_k$ is valid and saturated, and it is not an axiom, there exists a valid sequent $\Gamma_{k+1} \Rightarrow \Delta_{k+1}$ from which $\Gamma_k \Rightarrow \Delta_k$ is obtained by jump. We reason by contraposition. Let $\Gamma_k \Rightarrow \Delta_k$ be the saturated sequent $\Pi_1 \Rightarrow \Pi_2, [\Sigma_1 \lhd C_1], ..., [\Sigma_k \lhd C_k]$. Suppose that none of the sequents $C_1 \Rightarrow \Sigma_1, ..., C_k \Rightarrow \Sigma_k$ is valid. We prove that the sequent $\Gamma_k \Rightarrow \Delta_k$ is not valid.

By hypothesis there are models $\mathcal{M}_1, ..., \mathcal{M}_k$ falsifying sequents $C_1 \Rightarrow \Sigma_1, ..., C_k \Rightarrow \Sigma_k$. For $1 \leqslant j \leqslant k$, let $\mathcal{M}_j = \langle W_j, N^j, [\![\ ]\!]_j \rangle$ and for some elements $x_j \in W_j$ let $\mathcal{M}_j, x_j \Vdash C_j$ and $\mathcal{M}_j, x_j \nVdash S$ for all $S \in \Sigma_j$. Suppose all W_j are disjoint, i.e. $W_j \cap W_{j'} = \emptyset$. From these models we build a new model $\mathcal{M} = \langle W, N, [\![\ ]\!] \rangle$ as follows:

- $W = \bigcup W_l \cup \{x\}$, for x new;
- $N(z) = N^j(z)$, if $z \in W_j$;
- $N(x) = \{\alpha_1, ..., \alpha_k\}$, where $\alpha_k = \{x_k\}$; $\alpha_{k-1} = \{x_k, x_{k-1}\}$, ... , $\alpha_1 = \{x_k, ..., x_1\}$;
- $[\![p]\!] = \cup [\![p]\!]_j$, for p atomic and $p \in \Pi_2$;
- $[\![p]\!] = \cup [\![p]\!]_j \cup \{x\}$, for p atomic and $p \in \Pi_1$.

One can easily check that for E arbitrary formula or block, it holds that if $\mathcal{M}_j, x_j \Vdash E$, then $\mathcal{M}, x_j \Vdash E$, for $1 \leqslant j \leqslant k$.

To complete the proof we show that $\mathcal{M}$ falsifies each formula or block occurring in $\Gamma_k \Rightarrow \Delta_k$. Thus, we have to prove that

a) If $G \in \Gamma_k$, then $\mathcal{M}, x \Vdash G$, for G formula;
b) If $G \in \Delta_k$, then $\mathcal{M}, x \nVdash G$, for G formula;
c) If $[\Sigma_j \lhd A_j] \in \Delta_k$, then $\mathcal{M}, x \nVdash [\Sigma_j \lhd A_j]$.

The proof proceeds by induction on the modal degree of formulas. The base case and the inductive step for the propositional cases are immediate.
Proof of a). Let $G = C \preccurlyeq D$. For the saturation conditions ($\mathsf{com}^{\mathsf{i}}_{\mathsf{c}}$) and ($\mathsf{L} \preccurlyeq^{\mathsf{i}}$), it holds that for all blocks $[\Sigma_j \lhd A_j]$ in the saturated sequent, either $D \in \Sigma_j$ or there exists in the saturated sequent a block $[\Pi, \Sigma_l \lhd C]$, for $l \leqslant j$. Consider an arbitrary sphere $\alpha_j = \{x_k, ..., x_j\}$ and the corresponding block $[\Sigma_j \lhd A_j]$. There are two cases to consider: if i) $D \in \Sigma_j$, by construction of the model it holds that $\alpha_j \nVdash^{\exists} D$, i.e. $\alpha_j \Vdash^{\forall} \neg D$. Suppose that ii) there exists a block $[\Pi, \Sigma_l \lhd C]$ belonging to the saturated sequent $\Gamma_k \Rightarrow \Delta_k$. By construction of the model, we have that there exists a world x_l such that $x_l \Vdash C$; thus, $\alpha_l \Vdash^{\exists} C$. However, since the spheres are incremental, $\alpha_l \subseteq \alpha_j$; thus, $\alpha_j \Vdash^{\exists} C$. We have that for α_j arbitrary block, either $\alpha_j \Vdash^{\forall} \neg D$ or $\alpha_j \Vdash^{\exists} C$; thus, $\mathcal{M}, x \Vdash C \preccurlyeq D$.
Proof of b). Let $G = C \preccurlyeq D$. By the saturation condition ($\mathsf{R} \preccurlyeq^{\mathsf{i}}{}_{\mathsf{c}}$) there exists a block $[\Sigma_j \lhd A_j]$ belonging to $\Gamma_k \Rightarrow \Delta_k$ such that $C \in \Sigma_j$ and $D = A_j$. Let us consider $\alpha_j = \{x_k, ..., x_j\}$. We have that $C \in \Sigma_{j+1}, ..., C \in \Sigma_k$. By construction, $x_j \nVdash C$; therefore, $x_j \nVdash C, ..., x_k \nVdash C$. Furthermore, $x_j \Vdash A_j$; thus $x_j \Vdash D$. There exists $\alpha_j \in N(x)$ such that $\alpha_j \nVdash^{\forall} \neg D$ and $\alpha_j \nVdash^{\exists} C$; thus, $\mathcal{M}, x \nVdash C \preccurlyeq D$.
The proof of *c)* is similar to proof of *b)*.

We proved that if $\Gamma_k \Rightarrow \Delta_k$ is valid and saturated, and it is not an axiom, then there exists a valid sequent $\Gamma_{k+1} \Rightarrow \Delta_{k+1}$ from which $\Gamma_k \Rightarrow \Delta_k$ is obtained by jump. Since $md(\Gamma_{k+1} \Rightarrow \Delta_{k+1}) < md(\Gamma_k \Rightarrow \Delta_k)$, by inductive hypothesis we have that $\Gamma_{k+1} \Rightarrow \Delta_{k+1}$ is derivable; therefore, $\Gamma_k \Rightarrow \Delta_k$ is derivable as well, by the jump rule.

Completeness of $\mathcal{I}^{\mathsf{i}}_{\mathbb{VN}}$. If $md(\Gamma_0 \Rightarrow \Delta_0) = 0$, then any saturated sequent derived from it will have the form $\Gamma_k \Rightarrow \Delta_k, [\bot \lhd \top]$, where Γ_k and Δ_k are composed only of propositional formulas. If $\Gamma_k \Rightarrow \Delta_k$ is an axiom, we are done. If $\Gamma_k \Rightarrow \Delta_k$ is not an axiom, it has a propositional countermodel. Associate this countermodel to a world x, and build a model with $W = \{x\}$ and $N(x) = \{\{x\}\}$. It immediately follows that the model satisfies N. If $md(\Gamma_0 \Rightarrow \Delta_0) > 0$, the proof proceeds in the same way as for $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$. Notice that by inductive hypothesis all the models $\mathcal{M}_i$ involved in the construction satisfy N.

Completeness of $\mathcal{I}^{\mathsf{i}}_{\mathbb{VT}}$. We modify the definition of $N(x)$ in the model $\mathcal{M}$ by adding a new sphere α_0, in order to account for total reflexivity. Thus, $N(x) = \{\alpha_0, \alpha_1, \alpha_2, ..., \alpha_k\}$, where $\alpha_k = \{x_k\}$, $\alpha_{k-1} = \{x_k, x_{k-1}\}$, ..., $\alpha_1 = \{x_k, ..., x_1\}$, $\alpha_0 = \alpha_1 \cup \{x\}$. Cases $b)$ and $c)$ remain the same as in the completeness proof for $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$. As for $a)$, consider $N(x) = \{\alpha_0, \alpha_1, \alpha_2, ..., \alpha_k\}$. For spheres $\alpha_k, ..., \alpha_1$ $a)$ holds; we have to prove that also for α_0 either $\alpha_0 \Vdash^{\forall} \neg D$ or $\alpha_0 \Vdash^{\exists} C$. We know that either $i)$ $\alpha_1 \Vdash^{\forall} \neg D$ or $ii)$ $\alpha_1 \Vdash^{\exists} C$. If $i)$ holds, the theorem is proved, since $\alpha_0 \Vdash^{\exists} C$. If it holds that $(*)$ $\alpha_1 \nVdash^{\forall} \neg D$ then $ii)$ holds. By absurd, suppose $\alpha_0 \nVdash^{\forall} \neg D$; thus, $(**)$ $x \Vdash D$ (since all the other worlds did not satisfy D). By saturation condition (T^{i}), we have that either $D \in \Delta$ or $[\bot \lhd C] \in \Delta$. There are two cases to consider. If $D \in \Delta$, since $md(D) < md(C \preccurlyeq D)$, by inductive hypothesis we have $x \nVdash D$, against $(**)$. If $[\bot \lhd C] \in \Delta$, there exists a block $[\Sigma_u \lhd A_u]$ in the saturated sequent $\Gamma_k \Rightarrow \Delta_k$ such that $A_u = C$. Thus, by construction $\alpha_u \Vdash^{\exists} C$, and $x_u \Vdash C$ for some $x_u \in \alpha_u$. By construction $x_u \in \alpha_1$; thus, $\alpha_1 \Vdash^{\exists} C$ against $(*)$. We reached a contradiction; thus, also for α_0 it holds that $\alpha_0 \Vdash^{\forall} \neg D$ or $\alpha_0 \Vdash^{\exists} C$, and $\mathcal{M}, x \Vdash C \preccurlyeq D$.

Completeness of $\mathcal{I}^{\mathsf{i}}_{\mathbb{VW}}$. We modify $N(x)$ in order to account for weak centering by adding world x to each sphere, as follows: $N(x) = \{\alpha_1, \alpha_2, ..., \alpha_k\}$, where $\alpha_k = \{x_k, x\}$; $\alpha_{k-1} = \{x_k, x_{k-1}, x\}$, ..., $\alpha_1 = \{x_k, ..., x_1, x\}$. We have to prove that conditions $a)$, $b)$ and $c)$ hold, and the proof makes an essential use of the saturation condition (W^{i}).
Proof of a). By saturation condition ($\mathsf{L} \preccurlyeq^{\mathsf{i}}$) it holds that for all blocks $[\Sigma_j \lhd A_j]$ in the upper saturated sequent $D \in \Sigma_j$ or $A_j = C$. Let us consider $\alpha_j = \{x_k, ..., x_j, x\}$. There are two cases. If $i)$ $D \in \Sigma_j$, by construction we have that $x_u \nVdash D$ for $u = k, ..., j$ (refer to proof of $a)$ for $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$). We have to verify that the same holds for x. By the saturation condition (W^{i}), $D \in \Delta$; by inductive hypothesis, $x \nVdash D$; thus $\alpha_j \Vdash^{\forall} \neg D$. If $ii)$ $C = A_j$, then $x_j \Vdash C$, and $\alpha_j \Vdash^{\exists} C$. Thus, $\mathcal{M}, x \Vdash C \preccurlyeq D$.
Proof of b). By the saturation condition ($\mathsf{R} \preccurlyeq^{\mathsf{i}}$), there exists a block $[\Sigma_j \lhd A_j]$ such that $C \in \Sigma_j$ and $D = A_j$. Let us consider $\alpha_j = \{x_k, ..., x_j, x\}$. By construction it holds that $x_j \Vdash D$; thus, $\alpha_j \nVdash^{\forall} \neg D$. By construction it holds as well that $C \in \Sigma_k$; thus, $x_u \nVdash C$, for $u = k, ..., j$. We have to verify that also $x \nVdash C$. Since $C \in \Sigma_j$ and for saturation condition (W^{i}) it holds that $\Sigma_j \in \Delta$, and thus $C \in \Delta$. Apply the inductive hypothesis: since $md(C) < md(C \preccurlyeq D)$, $x \nVdash C$. Thus, $\alpha_j \nVdash^{\exists} C$ for each $C \in \Sigma_j$, and $\mathcal{M}, x \nVdash C \preccurlyeq D$.
Proof of c). We have to show that if $[\Sigma_j \lhd A_j] \in \Delta$, then $\mathcal{M}, x \nVdash [\Sigma_j \lhd A_j]$. Let us consider $\alpha_j = \{x_k, ..., x_j, x\}$. By construction it holds that $\alpha_j \nVdash^{\forall} \neg A_j$, and also that $x_u \Vdash E$ for each $E \in \Sigma_j$ and $u = k, ..., j$. By saturation condition (W^{i}), $\Sigma_j \subseteq \Delta$; for all $E \in \Sigma_j$ it holds that $x \nVdash E$ (refer to the previous case). Therefore $\alpha_j \nVdash^{\exists} E$ for each $E \in \Sigma_j$. Thus, it holds that $\mathcal{M}, x \nVdash [\Sigma_j \lhd A_j]$.

Completeness of $\mathcal{I}^{\mathsf{i}}_{\mathbb{VC}}$. For centering, we modify $N(x)$ by adding a new sphere α_{k+1}, which contains only x. Namely: $\alpha_{k+1} = \{x\}$; $\alpha_k = \{x_k, x\}$; $\alpha_{k-1} = \{x_k, x_{k-1}, x\}$,..., $\alpha_1 = \{x_k, ..., x_1, x\}$. Conditions $b)$ and $c)$ are as in the proof for $\mathcal{I}^{\mathsf{i}}_{\mathbb{VW}}$; case $a)$ is slightly different and employs the saturation condition (C^{i}). □

Completeness of calculi with absoluteness, i.e. $\mathcal{I}^{\mathsf{i}}_{\mathbb{VA}}$ and $\mathcal{I}^{\mathsf{i}}_{\mathbb{VNA}}$, need to follow a different procedure: Although rule A^{i} plays a similar role as jump, it does not reduce the modal degree when applied backwards. Thus, the proof strategy defined above would not apply.

In [35] completeness for these systems was proved by means of a translation to non-standard and complete calculi for the logics. Such a family of calculi was introduced by Lellmann and Pattinson (references: [53] and [52]) and covers all the systems of logic treated in the present section. The calculi $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$ can be simulated by the rules of the non-standard sequent calculi. Since for all the systems of non-standard calculi there is a cut-elimination proof, completeness of $\mathcal{I}^{\mathsf{i}}_{\mathbb{VA}}$ and $\mathcal{I}^{\mathsf{i}}_{\mathbb{VNA}}$ with respect to the axiom system follows. In order to make the present chapter more compact, we omit the non-standard calculi and the translation. Refer to [35] for a proof of this result. Moreover, we conjecture that internal and standard proof systems for $\mathbb{VA}$ and $\mathbb{VNA}$ could be defined by means of a different formalism, which would allow to have a direct proof of completeness with respect to the semantics. Some details are given in Section 4.11.

4.6 Logic $\mathbb{VTU}$ and extensions

Up to now, we have limited our attention to the basic logics in Lewis' family, i.e., systems $\mathbb{V}$, $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$, $\mathbb{VC}$, $\mathbb{VA}$ and $\mathbb{VTA}$. The question arises whether we can give internal and standard proof systems for the remaining systems of Lewis' family of conditional logics. In this section we treat the group of logics whose outer modal logic is **S5**: $\mathbb{VTU}$, $\mathbb{VWU}$, $\mathbb{VCU}$, $\mathbb{VTA}$, $\mathbb{VWA}$. We define proof systems for this family, including also $\mathbb{VCA}$, the stronger logic in Lewis' family, which collapses into classical logic. Let Strong range over the set $\{\mathbb{VTU}, \mathbb{VWU}, \mathbb{VCU}, \mathbb{VTA}, \mathbb{VWA}, \mathbb{VCA}\}$.

Several systems in Strong have an interest by themselves. According to Lewis, logic $\mathbb{VCU}$ is, along with $\mathbb{VC}$, one of the best candidate to formalize counterfactuals. Similarly as for $\mathbb{VC}$, a model for $\mathbb{VCU}$ is composed of a nested system of sphere, in which the smallest sphere, included in all the others, is a singleton containing only the actual world. The advantage of $\mathbb{VCU}$ is that the system satisfies the requirement of *universality*, that is, for all x, $\bigcup N(x) = W$[6]. Thus, there are no worlds occurring outside $\bigcup N(x)$ and, since by uniformity $\bigcup N(x)$ is the same for all the worlds $y \in W$, we can substitute W to $\bigcup N(x)$ when detailing the truth conditions on the model, thus simplifying the definitions. Logic $\mathbb{VCU}$ has a strong connection with belief update. Grahne established a precise mapping between belief update operators and Lewis' logic $\mathbb{VCU}$ [41]. The relation is expressed by the so-called *Ramsey's rule*:

$(A \circ B) \to C$ if and only if $A \to (B > C)$

where $\circ$ is any update operator satisfying Katsuno and Mendelzon's postulates [46]. The relation means that C is entailed by "A updated B" if and only if the conditional $B > C$ is entailed by A. In this sense, the conditional operator $B > C$ expresses a hypothetical update of a piece of information in A.

Furthermore, logic $\mathbb{VTU}$ and its extensions are equivalent to the logics of Comparative Concept Similarity studied in the context of ontologies [97]. These logics contain a connective $\leftleftarrows$, which allows to express sentences such as:

$$PicassoPainting \sqsubseteq BraquePainting \leftleftarrows GiottoPainting$$

asserting "Picasso's paintings are more similar to Braque's paintings than to Giotto's ones". The connective $\leftleftarrows$ can be seen as corresponding to the negation of $\preccurlyeq$. A semantic for the logic is provided in terms of Distance space models, defined as a set of worlds equipped with a distance function. It turns out that the basic logic of Comparative Concept Similarity coincides with Lewis' logic $\mathbb{VWU}$, while the one defined by "minspace" Distance models coincides with $\mathbb{VCU}$, so that Distance Space Models provide an alternative simple and natural semantics for conditional logics with uniformity [97, 4].

In order to define proof systems for $\mathbb{VTU}$ and its extensions it is convenient to abandon the nested sequent framework described in the previous section. Modal logic **S5** is the

[6] As seen in Chapter 1, this condition is equivalent to total reflexivity plus uniformity.

outer modal logic of Strong; thus, it is unlikely that a nested calculus with at most a level one of nesting, as the one defined for Base, would be the best option to describe such systems. We rather endorse the framework of *hypersequents*.

To the best of our knowledge, no other internal and standard proof systems for Lewis' logics with uniformity are known. Other than the labelled calculi presented in Chapter 3, a labelled system was given in [34], adopting a hybrid language and a relational semantics.

4.7 Hypersequent calculi $\mathcal{SH}^i_{\mathsf{Strong}}$

In order to define calculi for all the logics Strong we need to enrich the hypersequent framework with the following structural connectives: the conditional blocks $[\Sigma \lhd A]$ considered in section 4.2, encoding disjunctions of comparative plausibility formulas, and a second block structure $\langle\Theta\rangle$ encoding a "possible" formulas. The family of extended hypersequent calculi here presented are called $\mathcal{SH}^i$, for *standard hypersequent*.

Definition 4.7.1. A *conditional block* is a tuple $[\Sigma \lhd C]$ containing a multiset Σ of formulas and a single formula C. A *transfer block* is a multiset of formulas, written $\langle\Theta\rangle$. An *extended sequent* is a tuple $\Gamma \Rightarrow \Delta$ consisting of a multiset Γ of formulas and a multiset Δ containing formulas, conditional blocks, and transfer blocks. An *extended hypersequent* is a multiset of extended sequents, written $\Gamma_1 \Rightarrow \Delta_1 \Rightarrow \cdots \mid \Gamma_n \Rightarrow \Delta_n$. The *formula interpretation* of an extended sequent is (all blocks shown explicitly):

$$(\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd C_1], \ldots, [\Sigma_n \lhd C_n], \langle\Theta_1\rangle, \ldots, \langle\Theta_m\rangle)^{int} :=$$
$$\bigwedge \Gamma \rightarrow \bigvee \Delta \vee \bigvee_{B\in\Sigma_1} (B \preccurlyeq C_1) \vee \cdots \vee \bigvee_{B\in\Sigma_n} (B \preccurlyeq C_n) \vee \Diamond(\bigvee \Theta_1) \vee \cdots \vee \Diamond(\bigvee \Theta_m).$$

The *formula interpretation* of an extended hypersequent is given by

$$(\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n)^{int} \quad := \quad \Box(\Gamma_1 \Rightarrow \Delta_1)^{int} \vee \cdots \vee \Box(\Gamma_n \Rightarrow \Delta_n)^{int}$$

The rules of $\mathcal{SH}^i_{\mathsf{Strong}}$ are given in Figure 4.5. As with hypersequent calculi for modal logics, we can interpret each component of the hypersequent as encoding the informations relative to a world of the model. Rules $R \preccurlyeq^H$, $L \preccurlyeq^H$, jump^H and com^H treat conditional blocks. Observe that in $\mathcal{SH}^i_{\mathbb{VTU}}$ rule jump^H creates a new component of the hypersequent: thus, a new world. However, differently from what happened in $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$, the rule does not erase any informations in going from the conclusion to the premiss, due to the hypersequent structure. Thus, jump^H is now invertible.

The transfer block $\langle\Theta\rangle$ is needed to treat classes of comparative plausibility formulas, which become relevant with the condition of uniformity. As seen in Chapter 1, in models with uniformity (either with its local or global version) we have that for $x, y \in W$, $\bigcup N(x) = \bigcup N(y)$. Thus, in these models worlds are the same between all the systems of spheres, even if the spheres could be arranged differently in different systems. The outer sphere, corresponding to $\bigcup N(x)$, is the same among all the systems of spheres. Formulas $\perp \preccurlyeq \neg A$ and $\neg(\perp \preccurlyeq A)$, respectively corresponding to $\Box A$ and $\Diamond A$, are evaluated in $\bigcup N(x)$. As a consequence, these formulas are uniformly true in all worlds of $\bigcup N(x)$. For this reason, it is useful to include in the sequent calculus a specific structure to encode these formulas, which can "pass" between different worlds, i.e., between different components of the hypersequent. We have called such structures *transfer blocks*, and defined them as corresponding to a disjunction of possible formulas:

$$\langle\Theta\rangle = \neg(\perp \preccurlyeq T_1) \vee \cdots \vee \neg(\perp \preccurlyeq T_n) \qquad \textit{for } \Theta = T_1, \ldots, T_n$$

Initial sequents

$$\frac{}{\mathcal{G} \mid \Gamma, \bot \Rightarrow \Delta}\ \bot_{\mathsf{L}} \qquad\qquad \frac{}{\mathcal{G} \mid \Gamma, p \Rightarrow \Delta, p}\ \mathsf{init}$$

Propositional and conditional rules

$$\frac{\mathcal{G} \mid \Gamma, B \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, A}{\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta}\ \mathsf{L} \to^{H} \qquad\qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta, B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \to B}\ \mathsf{R} \to^{H}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B}\ \mathsf{R} \preccurlyeq^{H}$$

$$\frac{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd C]}{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C]}\ \mathsf{L} \preccurlyeq^{H}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}\ \mathsf{com}^{H}$$

$$\frac{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, \langle \Theta \rangle \mid A \Rightarrow \Theta \quad \mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, \langle \Theta, B \rangle}{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, \langle \Theta \rangle}\ \mathsf{T}^{H}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd A] \mid A \Rightarrow \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{jump}^{H} \qquad\qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \bot \rangle}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ \mathsf{in}_{\mathsf{trf}}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle \mid \Sigma \Rightarrow \Theta, \Pi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle \mid \Sigma \Rightarrow \Pi}\ \mathsf{jump}_{U} \qquad\qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle, \Theta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle}\ \mathsf{jump}_{T}$$

Rules for extensions

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd A], \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{W}^{H} \qquad\qquad \frac{\mathcal{G} \mid \Gamma, A \preccurlyeq B, A \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta}\ \mathsf{C}^{H}$$

$$\frac{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta \mid \Sigma, A \preccurlyeq B \mid \Pi}{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}\ \mathsf{abs}_{L} \qquad\qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow A \preccurlyeq B, \Delta \mid \Sigma \Rightarrow A \preccurlyeq B, \Pi}{\mathcal{G} \mid \Gamma \Rightarrow A \preccurlyeq B, \Delta \mid \Sigma \Rightarrow \Pi}\ \mathsf{abs}_{R}$$

$$\frac{\mathcal{G} \mid C, \Gamma \Rightarrow \Delta, [\Sigma \lhd C], \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd C]}\ \mathsf{jump}_{\mathsf{C}}$$

$$\mathcal{SH}^{i}_{\mathbb{VTU}} = \{\bot, \mathsf{init}, \mathsf{L} \to^{H}, \mathsf{R} \to^{H}\} \cup \{R \preccurlyeq^{H}, L \preccurlyeq^{H}, \mathsf{com}^{H}, \mathsf{jump}^{H}, \mathsf{T}^{H}, \mathsf{in}_{\mathsf{trf}}, \mathsf{jump}_{U}, \mathsf{jump}_{T}\}$$

$$\mathcal{SH}^{i}_{\mathbb{VWU}} = \mathcal{SH}^{i}_{\mathbb{VTU}} \cup \{\mathsf{W}^{H}\} \qquad \mathcal{SH}^{i}_{\mathbb{VCU}} = \mathcal{SH}^{i}_{\mathbb{VWU}} \cup \{\mathsf{C}^{H}\} \qquad \mathcal{SH}^{i}_{\mathbb{VTA}} = \mathcal{SH}^{i}_{\mathbb{VTU}} \cup \{\mathsf{abs}_{L}, \mathsf{abs}_{R}\}$$

$$\mathcal{SH}^{i}_{\mathbb{VWA}} = \mathcal{SH}^{i}_{\mathbb{VWU}} \cup \{\mathsf{abs}_{L}, \mathsf{abs}_{R}\} \qquad \mathcal{SH}^{i}_{\mathbb{VCA}} = \mathcal{SH}^{i}_{\mathbb{VCU}} \cup \{\mathsf{abs}_{L}, \mathsf{abs}_{R}, \mathsf{jump}_{\mathsf{C}}\}$$

Figure 4.5: Hypersequent calculi $\mathcal{SH}^{i}_{\mathsf{Strong}}$ for $\mathbb{VTU}$ and extensions

The rule T^H adds the consequent of comparative plausibility formulas to a transfer block; with rules jump_T and jump_U the "possible" formulas encoded by a transfer block pass to the same component and to all the other components of the hypersequent.

Rule $\mathsf{jump_C}$ is added to the rules of $\mathcal{SH}^i_{\mathbb{VCA}}$ to define a hypersequent system for $\mathbb{VCA}$. Recall that logic $\mathbb{VCA}$ collapses into classical logic, due to the combination of the conditions of absoluteness and centering. The former condition states that the systems of spheres associated to different worlds are the same, and the latter that for any world, the smaller sphere contains only that world. To satisfy both conditions, all the spheres and all the worlds collapse, and a model for $\mathbb{VCA}$ is composed of a single sphere around a single world - and this is a model for classical propositional logic. Thus, rule $\mathsf{jump_C}$ avoids introducing bottom-up any new component in the derivations - since the model is composed of just one world.

Example 4.7.1. Derivation of axiom (T) in $\mathcal{SH}^i_{\mathbb{VTU}}$.

$$\dfrac{\perp \preccurlyeq \neg A \Rightarrow \overset{\triangledown}{A}, \langle \perp \rangle \mid \perp \Rightarrow \perp \qquad \dfrac{\dfrac{A, \perp \preccurlyeq \neg A \Rightarrow \overset{\triangledown}{A}, \langle \perp, \neg A \rangle, \perp}{\perp \preccurlyeq \neg A \Rightarrow A, \langle \perp, \neg A \rangle, \perp, \neg A}\ \mathsf{R}\neg}{\perp \preccurlyeq \neg A \Rightarrow A, \langle \perp, \neg A \rangle}\ \mathsf{jump}_T}{\dfrac{\dfrac{\perp \preccurlyeq \neg A \Rightarrow A, \langle \perp \rangle}{\perp \preccurlyeq \neg A \Rightarrow A}\ \mathsf{in_{trf}}}{\Rightarrow \perp \preccurlyeq \neg A \to A}\ \mathsf{R}\to^H}\ \mathsf{T}^H$$

Example 4.7.2. Formula $A \preccurlyeq B \leftrightarrow (B \to A)$ is derivable in $\mathcal{SH}^i_{\mathbb{VCA}}$. Thus, in $\mathbb{VCA}$ the comparative plausibility operator collapses into classical material implication.

$$\dfrac{\dfrac{\dfrac{A, A \preccurlyeq \overset{\triangledown}{B}, B \Rightarrow A \qquad A \preccurlyeq B, \overset{\triangledown}{B} \Rightarrow A, B}{A \preccurlyeq B, B \Rightarrow A}\ \mathsf{C}^H}{A \preccurlyeq B \Rightarrow B \to A}\ \mathsf{R}\to^H}{\Rightarrow A \preccurlyeq B \to (B \to A)}\ \mathsf{R}\to^H$$

$$\dfrac{\dfrac{\dfrac{\dfrac{B, B \to A \Rightarrow \overset{\triangledown}{[A \lhd B]}, A, B \qquad A, B, B \to A \overset{\triangledown}{\Rightarrow} [A \lhd B], A}{B, B \to A \Rightarrow [A \lhd B], A}\ \mathsf{L}\to}{B \to A \Rightarrow [A \lhd B]}\ \mathsf{jump_C}}{B \to A \Rightarrow A \preccurlyeq B}\ \mathsf{R}\preccurlyeq^H}{\Rightarrow (B \to A) \to A \preccurlyeq B}\ \mathsf{R}\to^H$$

Remark 4.7.1. In [37] we used rule spl instead of $\mathsf{jump_C}$ to define $\mathcal{SH}^i_{\mathbb{VCA}}$.

$$\dfrac{\mathcal{G} \mid \Sigma, \Gamma \Rightarrow \Pi, \Delta}{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta}\ \mathsf{spl}$$

The rule has the same function as $\mathsf{jump_C}$: looking at the rule bottom-up, any two component of the hypersequent can be united into one component. This rule is less economical than $\mathsf{jump_C}$: $\mathcal{SH}^i_{\mathbb{VCA}}$ derivations using spl would need to introduce bottom-up new components and then unify them, while derivations with $\mathsf{jump_C}$ avoid their introduction in the first place.

Theorem 4.7.1 (Soundness)**.** The rules of $\mathcal{SH}^i_{\mathsf{Strong}}$ are sound with respect to the formula interpretation *int*. Hence the calculi are sound for the respective logics, i.e.: If $\mathcal{SH}^i_{\mathsf{Strong}} \vdash \mathcal{G}$, then $(\mathcal{G})^{int}$ is a theorem of the corresponding logic.

Proof. For each rule, the proof strategy consists in assuming that the premiss(es) are valid in all worlds of all models, and assume that there is a model and a world in which the conclusion is not valid. If we reach a contradiction, we have a proof that the sequent calculus rules preserve validity and that, consequently, they are sound.

The cases for propositional rules are standard. For the rules $\mathsf{R} \preccurlyeq^H$, $\mathsf{L} \preccurlyeq^H$, com^H, jump^H, W^H, C^H the proof is an easy extension of the argument in Section 4.2, taking into account that $\Box$ is transitive in models for $\mathbb{VTU}$. For $\mathsf{in}_{\mathsf{trf}}$ the proof is trivial. For abs_L and abs_R soundness follows from absoluteness of the accessibility relation in extensions of $\mathbb{VTA}$.

For T^H, suppose that for $\mathcal{M} = \langle W, N, [\![\ ]\!]\rangle$ we have:

1. $\mathcal{M}, x \Vdash (\mathcal{G})^{int} \vee \Box(\Gamma \Rightarrow \Delta, \langle\Theta\rangle)^{int} \vee \Box\,(A \Rightarrow \Theta)^{int}$;
2. $\mathcal{M}, x \Vdash (\mathcal{G})^{int} \vee \Box\,(\Gamma \Rightarrow \Delta, \langle\Theta, B\rangle)^{int}$;
3. $\mathcal{M}, x \nVdash (\mathcal{G})^{int} \vee \Box\,(\Gamma, A \preccurlyeq B \Rightarrow \Delta, \langle\Theta\rangle)^{int}$.

From 3 we have that $x \nVdash (\mathcal{G})^{int}$ and that $x \nVdash \Box((\bigwedge\Gamma \wedge A \preccurlyeq B) \to (\bigvee\Delta \vee \Diamond\bigvee\Theta))$, thus $x \Vdash \Diamond\neg((\bigwedge\Gamma \wedge A \preccurlyeq B) \to (\bigvee\Delta \vee \Diamond\bigvee\Theta))$. This means that there exists $\alpha \in N(x)$ and an $y \in \alpha$ such that

4. $y \Vdash F$ for all $F \in \Gamma$;
5. $y \Vdash A \preccurlyeq B$, which means that for all $\alpha \in N(y), \alpha \Vdash^{\forall} \neg B$ or $\alpha \Vdash^{\exists} A$;
6. $y \nVdash G$ for all $G \in \Delta$;
7. $y \nVdash \Diamond\bigvee\Theta$, i.e. $y \Vdash \Box\neg\bigvee\Theta$: for all $w \in \bigcup N(y)\,.\,w \nVdash Q$ for all $Q \in \Theta$.

Consider 2: since $x \nVdash (\mathcal{G})^{int}$, it must hold that $x \Vdash \Box\,(\Gamma \Rightarrow \Delta, \langle\Theta, B\rangle)^{int}$. The valid formula is boxed, and thus its content should apply to all elements belonging to $\bigcup N(x)$. Since $y \in \alpha$ and $\alpha \in N(x)$, we have $y \in \bigcup N(x)$. We are therefore justified to apply 2 to y, and from 2, 4 and 6 we obtain that $y \Vdash \Diamond(\bigvee\Theta \vee B)$. Thus, there exists $z \in \bigcup N(y)$ such that $z \Vdash Q$ for some $Q \in \Theta$ or $z \Vdash B$. The first disjunct contradicts with 7, thus

8. $z \Vdash B$.

Now consider 1: its first and second disjunct cannot hold. Thus, $x \Vdash \Box(A \to \bigvee\Theta)^{int}$ holds. This means that for all the elements w occurring in $\bigcup N(x)$ it holds that $w \Vdash \neg A$ or $w \Vdash Q$ for some $Q \in \Theta$. By uniformity we have that $\bigcup N(x) = \bigcup N(y)$. From 7 we have that the second disjunct cannot hold; thus

9. $w \Vdash \neg A$.

Finally, consider 5: its first disjunct is in contradiction with 8, and its second disjunct is in contradiction with 9.

For jump_U, suppose

1. $\mathcal{M}, x \Vdash (\mathcal{G})^{int} \vee \Box\,(\Gamma \Rightarrow \Delta, \langle\Theta\rangle)^{int} \vee \Box\,(\Sigma \Rightarrow \Theta, \Pi)^{int}$
2. $\mathcal{M}, x \nVdash (\mathcal{G})^{int} \vee \Box\,(\Gamma \Rightarrow \Delta, \langle\Theta\rangle)^{int} \vee \Box\,(\Sigma \Rightarrow \Pi)^{int}$.

From 2 we have:

3. $x \nVdash (\mathcal{G})^{int}$;
4. $x \nVdash \Box(\bigwedge\Gamma \to (\bigvee\Delta \vee \Diamond\bigvee\Theta))$;
5. $x \nVdash \Box(\bigwedge\Sigma \to \bigvee\Pi)$.

Form 4, there exists $y \in \bigcup N(x)$ such that

6. $y \Vdash F$ for all $F \in \Gamma$;
7. $y \nVdash G$ for all $G \in \Delta$;
8. $y \nVdash \Diamond\bigvee\Theta$, i.e., for all $w \in \bigcup N(y)$, $w \nVdash Q$ for all $Q \in \Theta$.

Similarly, from 5 we have that there exists $z \in \bigcup N(x)$ such that

9. $z \Vdash S$ for all $S \in \Sigma$;
10. $z \nVdash P$ for all $P \in \Pi$.

From 1 we have that $x \Vdash (\mathcal{G})^{int}$ or $x \Vdash \Box(\bigwedge\Gamma \to \bigvee\Delta \vee \Diamond\bigvee\Theta)$ or that $x \Vdash \Box(\bigwedge\Sigma \Rightarrow \bigvee\Theta \vee \bigvee\Pi)$. The first disjunct contradicts with 3, and the second with 6, 7 and 8, observing that by uniformity $\bigcup N(y) = \bigcup N(x)$. From the third disjunct we have that for all $w \in \bigcup N(x)$ either $w \nVdash S$ for some $S \in \Sigma$, which contradicts with 9, or $w \Vdash P$

for some $P \in \Pi$, which contradicts with 10, or $w \Vdash Q$ for some $Q \in \Theta$. By uniformity, $\bigcup N(y) = \bigcup N(x)$, and the latter disjunct contradicts with 8.
For jump_T, suppose

1. $\mathcal{M}, x \Vdash (\mathcal{G})^{int} \vee \Box\, (\Gamma \Rightarrow \Delta, \langle\Theta\rangle\, , \Theta)^{int}$;
2. $\mathcal{M}, x \nVdash (\mathcal{G})^{int} \vee \Box\, (\Gamma \Rightarrow \Delta, \langle\Theta\rangle)^{int}$.

From 2 we have that $x \nVdash (\mathcal{G})^{int}$ and that $x \nVdash \Box(\bigwedge\Gamma \to (\bigvee\Delta \vee \Diamond\bigvee\Theta))$, i. e. $x \Vdash \Diamond\neg(\bigwedge\Gamma \to (\bigvee\Delta \vee \Diamond\bigvee\Theta))$. This means that there exists $y \in \bigcup N(x)$ such that

3. $y \Vdash F$ for some $F \in \Gamma$;
4. $y \nVdash G$ for some $G \in \Delta$;
5. $y \nVdash \Diamond\bigvee\Theta$, meaning that for all $z \in \bigcup N(y)$, $z \nVdash Q$ for all $Q \in \Theta$.

As for 1, its first disjunct cannot hold; thus, $x \Vdash \Box(\bigwedge\Gamma \to (\bigvee\Delta \vee \Diamond\bigvee\Theta \vee \bigvee\Theta))$ holds. Then, for all $w \in \bigcup N(x)$ it holds that $w \nVdash F$ for some $F \in \Gamma$ or there exists $k \in \bigcup N(w)$ such that $k \Vdash Q$ for some $Q \in \Theta$, or $w \Vdash G$ for some $G \in \Delta$ or $w \Vdash Q$ for some $Q \in \Theta$. The first and second disjunct contradict with 3, and 4 respectively, since by uniformity $\bigcup N(x) = \bigcup N(y)$; the third and fourth disjunct contradicts with 5, since $z \in \bigcup N(x) = \bigcup N(y)$.
For jump_C, suppose

1. $\mathcal{M}, x \Vdash (\mathcal{G})^{int} \vee \Box(C, \Gamma \Rightarrow \Delta, [\Sigma \lhd C]\, , \Sigma)^{int}$;
2. $\mathcal{M}, x \nVdash (\mathcal{G})^{int} \vee \Box\, (\Gamma \Rightarrow \Delta, [\Sigma \lhd C])^{int}$.

From 2 we have that $x \nVdash (\mathcal{G})^{int}$ and $x \nVdash \Box(\bigwedge\Gamma \to \bigvee\Delta \vee \bigvee_{B\in\Sigma} B \preccurlyeq C)$. Thus, there exists a $y \in \bigcup N(x)$ such that $y \Vdash \bigwedge\Gamma$, $y \nVdash \bigvee\Delta$, and $y \nVdash \bigvee_{B\in\Sigma} B \preccurlyeq C$. From this latter, we obtain that for all $B \in \Sigma$ there exists a neighbourhood $\alpha \in N(y)$ such that $\alpha \Vdash^{\exists} C$ and $\alpha \nVdash^{\exists} B$, and thus that there exists a $k \in \alpha$ such that $k \Vdash C$ and $k \nVdash B$. Since the model satisfies both absoluteness and centering, it is composed of just one world, and all the worlds k introduced are such that $k = x$. Thus we have that $x \Vdash C$ and $x \nVdash B$, for all $B \in \Sigma$; this contradicts with $x \nVdash C$ and $x \Vdash B$ for at least one formula $B \in \Sigma$, that can be obtained from 1. $\square$

We do not have a syntactic proof of cut elimination for $\mathcal{SH}^i_{\mathsf{Strong}}$. Nevertheless, the rules of weakening and contraction are admissible in the calculi, and all the rules are invertible.

Lemma 4.7.2. The rules of weakening Wk, $\mathsf{Wk_E}$, $\mathsf{Wk_i}$, $\mathsf{Wk_B}$ and $\mathsf{Wk_T}$ are height-preserving admissible in $\mathcal{SH}^i_{\mathsf{Strong}}$.

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Sigma, \Gamma \Rightarrow \Delta, \Pi}\ \mathsf{Wk} \qquad\qquad \frac{\mathcal{G}}{\mathcal{G} \mid \Rightarrow}\ \mathsf{Wk_E}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd C]}\ \mathsf{Wk_B} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd C]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, \Omega \lhd C]}\ \mathsf{Wk_i} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle\Theta\rangle}\ \mathsf{Wk_T}$$

Proof. The proof is by mutual inductive on the height of a derivation and does not present particular difficulties. $\square$

Lemma 4.7.3. All rules of $\mathcal{SH}^i_{\mathsf{Strong}}$ are height-preserving invertible.

Proof. For all the rules (R) it holds that, if the conclusion of (R) is derivable, so are the premisses. The proof proceeds by induction on the height of the derivation. Invertibility of propositional rules and $\mathsf{R} \preccurlyeq^H$ follows from application of the inductive hypothesis to the rule used to derive the conclusion of (R). Invertibility of all the other rules - including jump^H - follows from admissibility of the rules of weakening. $\square$

Lemma 4.7.4. The rules of contraction $\mathsf{Ctr_L}, \mathsf{Ctr_R}, \mathsf{Ctr_B}, \mathsf{Ctr_i}$ are height-preserving admissible in $\mathcal{SH}^i_{\mathsf{Strong}}$.

$$\frac{\mathcal{G} \mid A, A, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid A, \Gamma \Rightarrow \Delta}\ \mathsf{Ctr_L} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A, A}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A}\ \mathsf{Ctr_R} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd A]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{Ctr_B}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, A, A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, A \lhd B]}\ \mathsf{Ctr_i} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta, A, A \rangle}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta, A \rangle}\ \mathsf{Ctr_T}$$

Proof. Admissibility of $\mathsf{Ctr_L}$ and $\mathsf{Ctr_R}$ proceeds by induction on the height of the derivation. The proofs of admissibility of $\mathsf{Ctr_B}$ and $\mathsf{Ctr_T}$ are also proved by induction on the height of the derivation, and make use of admissibility of $\mathsf{Ctr_L}$, $\mathsf{Ctr_R}$. By means of example, let us consider the case of $\mathsf{Ctr_B}$ when $\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma, A, A \lhd B]$ is obtained by an application of jump^H from $\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma, A, A \lhd B] \mid B \Rightarrow \Sigma, A, A$. By inductive hypothesis, we have a derivation of $\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma, A \lhd B] \mid B \Rightarrow \Sigma, A, A$. Admissibility of $\mathsf{Ctr_R}$ yields $\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma, A \lhd B] \mid B \Rightarrow \Sigma, A$, to which we apply jump^H to obtain a derivation of $\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma, A \lhd B]$. The other cases are similar. The proof of admissibility of $\mathsf{Ctr_i}$ also is by inductive on the height of the derivation, using admissibility of $\mathsf{Wk_i}$ if the last applied rule was $\mathsf{L} \preccurlyeq^H$ or com^H. □

4.8 Semantic completeness of $\mathcal{SH}^i_{\mathbb{VTU}}$, $\mathcal{SH}^i_{\mathbb{VWU}}$ and $\mathcal{SH}^i_{\mathbb{VCU}}$

In this section we prove completeness for calculi $\mathcal{SH}^i_{\mathbb{VTU}}$, $\mathcal{SH}^i_{\mathbb{VWU}}$ and $\mathcal{SH}^i_{\mathbb{VCU}}$. Since we have no syntactic cut-elimination for these calculi, proof of their completeness proceeds semantically, by showing that a countermodel can be constructed from a branch of failed derivation tree. A proof of completeness of the hypersequent calculi for $\mathbb{VTA}$, $\mathbb{VWA}$ and $\mathbb{VCA}$ (the systems with absoluteness) can be found in [37]. The article presents a family of non-standard and cut-free hypersequent calculi, equivalent to $\mathcal{SH}^i_{\mathsf{Strong}}$. Completeness of all the standard proof systems can be proved via the translation to the non-standard calculi. Instead of referring to the non-standard systems, in next section we present standard proof systems to $\mathcal{SH}^i_{\mathbb{VTA}}$ and extensions, and prove completeness for that system[7].

The present proof proceeds similarly to the proof of semantic completeness for $\mathbb{V}$ and extensions in Section 4.5. We consider a cumulative version of the hypersequent calculi, in which some rules are modified as follows.

$$\frac{\mathcal{G} \mid \Gamma, A \to B, B \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta, A}{\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta}\ \mathsf{L} \to^H_c$$

$$\frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta, A \to B, B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \to B}\ \mathsf{R} \to^H_c \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B, [A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B}\ \mathsf{R} \preccurlyeq^H_c$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_1 \lhd A], [\Sigma_2 \lhd B] \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd B] [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}\ \mathsf{com}^H_c$$

Definition 4.8.1. An extended hypersequent $\mathcal{G}$ is $\mathbb{VTU}$-*saturated* if it is not an instance of init or $\bot_\mathsf{L}$, and it satisfies all of the following conditions:

1. $(\mathsf{L} \to^H_c)$ if $\Gamma, A \to B \Rightarrow \Delta \in \mathcal{G}$, then $B \in \Gamma$ or $A \in \Delta$;
2. $(\mathsf{R} \to^H_c)$ if $\Gamma \Rightarrow A \to B, \Delta \in \mathcal{G}$, then $A \in \Gamma$ and $B \in \Delta$;

[7] It should be possible to prove semantic completeness also for $\mathcal{SH}^i_{\mathbb{VTA}}$, $\mathcal{SH}^i_{\mathbb{VWA}}$ and $\mathcal{SH}^i_{\mathbb{VCA}}$. However, the proof would be quite complex; for this reason we prefer to give an alternative proof system.

3. $(\mathsf{R} \preccurlyeq_c^H)$ if $\Gamma \Rightarrow \Delta, A \preccurlyeq B \in \mathcal{G}$, then $[\Sigma, A \lhd B] \in \Delta$ for some Σ;
4. $(L \preccurlyeq^H)$ if $\Gamma, C \preccurlyeq D \Rightarrow \Delta, [\Sigma \lhd A] \in \mathcal{G}$, then $D \in \Sigma$ or $[\Sigma \lhd C] \in \Delta$;
5. (com_c^H) if $\Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Pi \lhd B] \in \mathcal{G}$, then $\Sigma \subseteq \Pi$ or $\Pi \subseteq \Sigma$;
6. (jump^H) if $\Gamma \Rightarrow \Delta, [\Sigma \lhd A] \in \mathcal{G}$, then $A, \Theta \Rightarrow \Sigma, \Pi \in \mathcal{G}$ for some Θ, Π;
7. (T^H) if $\Gamma, C \preccurlyeq D \Rightarrow \Delta, \langle\Theta\rangle \in \mathcal{G}$, then $D \in \Theta$ or $C, \Sigma \Rightarrow \Theta, \Pi \in \mathcal{G}$ for some Σ, Π;
8. $(\mathsf{in}_{\mathsf{trf}})$ if $\Gamma \Rightarrow \Delta \in \mathcal{G}$, then $\langle\Theta\rangle \in \Delta$ for some Θ;
9. $(\mathsf{jump}_U^i, \mathsf{jump}_T^i)$ if $\Gamma \Rightarrow \Delta, \langle\Theta\rangle \in \mathcal{G}$ and $\Sigma \Rightarrow \Pi \in \mathcal{G}$, then $\Theta \subseteq \Pi$.

It is $\mathbb{VWU}$-*saturated* (resp. $\mathbb{VCU}$-*saturated*) if it also satisfies (W^H) (resp. (C^H)) below:

1. (W^H) if $\Gamma \Rightarrow \Delta, [\Sigma \lhd A] \in \mathcal{G}$, then $\Sigma \subseteq \Delta$;
2. (C^H) if $\Gamma, C \preccurlyeq D \Rightarrow \Delta \in \mathcal{G}$, then $C \in \Gamma$ or $D \in \Delta$;

We construct a countermodel from a $\mathbb{VTU}$-saturated extended hypersequent $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ as follows:

- $W := \{1, \ldots, n\}$
- $[\![p]\!] := \{i \leq n : p \in \Gamma_i\}$

The sphere systems $N(i)$ for $i \leq n$ are then defined as follows: Assume that $\Gamma_i \Rightarrow \Delta_i$ is sequent

$$\Gamma_i \Rightarrow \Delta_i', [\Sigma_1 \lhd A_1], \ldots, [\Sigma_k \lhd A_k]$$

where Δ_i' contains no conditional blocks. Due to saturation condition (com^H) we may assume that $\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots \subseteq \Sigma_k$. Moreover, by condition (jump^H) for every $j \leq k$ there is a component $\Gamma_{m_j} \Rightarrow \Delta_{m_j} \in \mathcal{G}$ with $A_j \in \Gamma_{m_j}$ and $\Sigma_j \subseteq \Delta_{m_j}$. Hence we set

$$N(i) := \{\{m_k\}, \{m_k, m_{k-1}\}, \ldots, \{m_k, \ldots, m_1\}, W\}$$

Call the resulting structure $\mathcal{M}_\mathcal{G}$.

Lemma 4.8.1. For a $\mathbb{VTU}$-saturated hypersequent $\mathcal{G}$ the structure $\mathcal{M}_\mathcal{G}$ is a $\mathbb{VTU}$-model.

Proof. Nesting of spheres is obvious from the fact that $\{m_k\} \subseteq \{m_k, m_{k-1}\} \subseteq \cdots \subseteq \{m_k, \ldots, m_1\} \subseteq W$; reflexivity and uniformity follow from the fact that $W \in N(i)$. $\square$

Definition 4.8.2. The complexity of a formula $A \in \mathcal{F}^{\preccurlyeq}$ is defined as follows:

- $w(P) = w(\bot) = 1$;
- $w(A \circ B) = w(A) + w(B) + 1$ for $\circ$ implication and comparative plausibility.

Lemma 4.8.2. Let $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ be a $\mathbb{VTU}$-saturated extended hypersequent and let $\mathcal{M}_\mathcal{G}$ be defined as above with world i associated to component $\Gamma_i \Rightarrow \Delta_i$. Then:

1. given a formula A, if $A \in \Gamma_i$ then $\mathcal{M}_\mathcal{G}, i \Vdash A$
2. given a formula A, if $A \in \Delta_i$ then $\mathcal{M}_\mathcal{G}, i \nVdash A$
3. given a block $[\Sigma \lhd C]$, if $[\Sigma \lhd C] \in \Delta_i$, then $\mathcal{M}_\mathcal{G}, i \nVdash \bigvee_{B \in \Sigma}(B \preccurlyeq C)$
4. given a formula B, if $\langle\Theta, B\rangle \in \Delta_i$ for some Θ, then $\mathcal{M}_\mathcal{G}, i \nVdash \Diamond B$.

Proof. We prove statements 1 and 2 by mutual inductive on the complexity of A. The base case and the propositional cases are straightforward, hence we consider $A = E \preccurlyeq F$. Let $i \in W$ be the world associated to $\Gamma_i \Rightarrow \Delta_i$, with

$$\Delta_i = \Delta_i', [\Sigma_1 \lhd D_1], \ldots, [\Sigma_k \lhd D_k], \langle\Theta\rangle$$

where Δ_i' contains no conditional block and $\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots \subseteq \Sigma_k$.

1. Suppose $E \preccurlyeq F \in \Gamma_i$, and let $\alpha \in N(i)$. We have to show that either $\alpha \nVdash^{\exists} F$ or $\alpha \Vdash^{\exists} E$. We distinguish the case where $\alpha = \{m_k, \dots, m_j\}$ for some $j \leq k$ and $\alpha = W$. In the former case to each $m_l \in \alpha$ is associated to a sequent

$$D_l, \Lambda_l \Rightarrow \Pi_l, \Sigma_l$$

which belongs to $\mathcal{G}$ by the saturation condition (jump^H) applied to a conditional block $[\Sigma_l \lhd D_l] \in \Delta_i$, for $l = j, \dots, k$. Observe that by saturation condition ($L \preccurlyeq^H$), either $F \in \Sigma_j$ or $E = D_j$. Since $w(F) < w(E \preccurlyeq F)$, by inductive hypothesis we obtain $\mathcal{M}_{\mathcal{G}}, m_j \nVdash F$. Then, since $\Sigma_j \subseteq \cdots \subseteq \Sigma_k$, we also have $F \in \Sigma_l$ for $l = j, \dots, k$, whence by inductive hypothesis we get $\mathcal{M}_{\mathcal{G}}, m_l \nVdash F$, for $l = j, \dots, k$. Thus, $\alpha \Vdash^{\forall} \neg F$, i.e., $\alpha \nVdash^{\exists} F$. Now let $E = D_m$, for some $j \leqslant m \leqslant k$. Since $w(E) < w(E \preccurlyeq F)$, by inductive hypothesis on the sequent $E, \Lambda_m \Rightarrow \Pi_m, \Sigma_m$, we obtain $\mathcal{M}_{\mathcal{G}}, m_j \Vdash E$, and thus $\alpha \Vdash^{\exists} E$.

   Now let $\alpha = W$. Since $E \preccurlyeq F \in \Gamma_i$, by saturation condition (T) either $F \in \langle\Theta\rangle$, or for some Λ_j, Π, $E, \Lambda \Rightarrow \Pi, \Theta \in \mathcal{G}$. In the latter case there is a world j associated to the sequent $E, \Lambda \Rightarrow \Pi, \Theta \in \mathcal{G}$, and by inductive hypothesis on E, we get $\mathcal{M}_{\mathcal{G}}, j \Vdash E$, whence $W \Vdash^{\exists} E$. In the former case let us consider any $k \in W$ (including $k = i$). To k is associated a seguent $\Gamma_k \Rightarrow \Delta_k$, and by saturation conditions (jump_T) and (jump_U) we have $\Theta \subseteq \Delta_k$, whence $F \in \Delta_k$. By inductive hypothesis on F we conclude $\mathcal{M}_{\mathcal{G}}, k \nVdash F$. This shows that for all $y \in W$, $y \nVdash F$, whence $\alpha \nVdash^{\exists} F$.

2. Let $E \preccurlyeq F \in \Delta'_i$. We have to show that there exists a sphere $\alpha \in N(i)$ such that $\alpha \Vdash^{\exists} F$ and $\alpha \nVdash^{\exists} E$. Recall that $N(i) = \{\{m_k\}, \{m_k, m_{k-1}\}, \dots, \{m_k, \dots, m_1\}, W\}$ where each m_l is associated to a sequent $D_l, \Lambda_l \Rightarrow \Pi_l, \Sigma_l \in \mathcal{G}$ coming from a block $[\Sigma_l \lhd D_l] \in \Delta_i$, for $l = j, \dots, k$. By saturation, there is some $j \leq k$ such that $D_j = F$ and $E \in \Sigma_j$. Let us consider world m_j associated to the sequent $F, \Lambda_j \Rightarrow \Sigma_j, \Pi \in \mathcal{G}$. By inductive hypothesis applied to F and $F, \Lambda_j \Rightarrow \Sigma_j, \Pi$, we get $\mathcal{M}_{\mathcal{G}}, m_j \Vdash F$. Similarly by inductive hypothesis we get $\mathcal{M}_{\mathcal{G}}, m_j \nVdash E$. Since $\Sigma_j \subseteq \cdots \subseteq \Sigma_k$, we also get $\mathcal{M}_{\mathcal{G}}, m_l \nVdash E$, for $l = j, \dots, k$. Thus if we consider the sphere $\alpha = \{m_k, \dots, m_j\} \in N(i)$, we obtain that $\alpha \Vdash^{\exists} F$ and $\alpha \nVdash^{\exists} E$.

The proof of 3 uses 2, recalling that a block is a disjunction of $\preccurlyeq$-formulas. The proof of 4 uses 2 with an argument as in the proof of 1 for the case of $\alpha = W$ with $B \in \langle\Theta\rangle$. $\square$

The countermodel construction described above can be extended to $\mathbb{VWU}$ and $\mathbb{VCU}$ by modifying the definition of the model as follows. For $\mathbb{VWU}$, let

$$N(i) := \{\{m_k, i\}, \{m_k, m_{k-1}, i\}, \dots, \{m_k, \dots, m_1, i\}, W\}.$$

For $\mathbb{VCU}$, we add $\{i\}$ to $N(i)$ for any i. The proof of Lemma 4.8.2 can be easily extended to both cases (statements 1 and 2), using the specific saturation conditions for these systems. From Lemma 4.8.2 we obtain:

Lemma 4.8.3. For $\mathcal{L} \in \{\mathbb{VTU}, \mathbb{VWU}, \mathbb{VCU}\}$ let $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ be a $\mathcal{L}$-saturated hypersequent and let $\mathcal{M}_{\mathcal{G}}$ be defined as above, then

- for any $i \in W$ associated to sequent $\Gamma_i \Rightarrow \Delta_i$ we have $\mathcal{M}_{\mathcal{G}}, i \nVdash (\Gamma_i \Rightarrow \Delta_i)^{int}$;
- for any $i \in W$ we have $\mathcal{M}_{\mathcal{G}}, i \nVdash (\mathcal{G})^{int}$.

To use these results in a decision procedure, we apply a strategy of *local loop checking*: rules are not applied if there is a premiss from which the conclusion is derivable using structural rules. Since these are all admissible in $\mathcal{SH}^c_{\mathcal{L}}$, this does not jeopardise completeness.

Proposition 4.8.4. Backwards proof search with local loop checking terminates and every leaf of the resulting derivation is an axiom or a saturated hypersequent.

Proof. By Lemmas 4.7.2 and 4.7.4, we may assume that the proof search only considers *duplication-free* sequents, i.e., sequents containing duplicates neither of formulas nor of blocks. By the subformula property, the number of duplication-free sequents possibly relevant to a derivation of a sequent is bounded in the number of subformulas of that sequent, and hence backwards proof search for $\mathcal{G}$ terminates. Furthermore, every leaf is either an axiom or a saturated sequent, since otherwise another rule could be applied. $\square$

Theorem 4.8.5 (Completeness)**.** If $(\mathcal{G})^{int}$ is a theorem of logics $\mathcal{L}$, then $\mathcal{SH}^i_{\mathcal{L}} \vdash \mathcal{G}$, for $\mathcal{L} \in \{\mathbb{VTU}, \mathbb{VWU}, \mathbb{VCU}\}$.

Proof. By Proposition 4.8.4 backwards proof search with root $\mathcal{G}$ terminates and every leaf of it is an axiom or a saturated sequent. By invertibility of the rules each sequent $\mathcal{G}'$ occurring as a leaf is valid. In this case $\mathcal{G}'$ must be an axiom, since otherwise, by Lem. 4.8.3 we can bulid a countermodel $\mathcal{M}_{\mathcal{G}'}$ falsifying $(\mathcal{G}')^{int}$ and hence by monotonicity also $(\mathcal{G})^{int}$. $\square$

Proposition 4.8.4 gives rise to a (non-optimal) CO-NEXPTIME-decision procedure for validity. Root-first proof search with local loop checking to an input sequent $\Rightarrow G$ terminates and every leaf of the resulting derivation is an instance of init or $\bot_{\mathsf{L}}$ or a saturated sequent. Thus, in order to check whether $\Rightarrow G$ is derivable is suffices to non-deterministically choose a duplication-free $\mathcal{L}$-saturated extended hypersequent containing only subformulas of G and containing a component $\Gamma \Rightarrow \Delta, G$. If this is not possible, then backwards proof search will produce a proof of $\Rightarrow G$. But if it is possible, then by Lemma 4.8.3 this hypersequent gives rise to a countermodel for G. Since the size of duplication-free extended hypersequents consisting of subformulas of G is bounded exponentially in the number of subformulas of G, this gives the CO-NEXPTIME complexity bound. This result is not optimal, since it is known that the logics of this section are EXPTIME-complete [28].

4.9 Head-hypersequents calculi $\mathcal{HH}^i_{\mathsf{Strong}^{\mathsf{A}}}$

The hypersequent calculus $\mathcal{SH}^i_{\mathbb{VTU}}$ presented in Section 4.7 are specifically tailored to capture conditional logics with total reflexivity and uniformity. We have modularly extended the calculi to extensions of $\mathbb{VTU}$; however, for the logics with total reflexivity and absoluteness it is possible to define a slightly different proof system - simpler and more adapted to the structure of the models.

The semantic condition corresponding to systems with absoluteness is that any two systems of spheres are equal: for $x, y \in W$, $N(x) = N(y)$. As a consequence, the comparative plausibility formulas which are true at a system of spheres (and thus: at a world) are true at all the other systems of spheres (and thus at all the worlds). Absoluteness implies uniformity, since this latter condition requires only the union on any two sets of sphere to be equivalent. With uniformity, only "possible" formulas are the same between all the worlds (refer to Section 4.7). If we have absoluteness, all comparative plausibility formulas - and not just possible formulas - are allowed to "pass" between the worlds.

Let $\mathsf{Strong}^{\mathsf{A}}$ range over the set $\{\mathbb{VTA}, \mathbb{VWA}, \mathbb{VCA}\}$. To define proof systems for $\mathsf{Strong}^{\mathsf{A}}$ we no longer need transfer blocks; we instead enrich the structure of hypersequents by adding an additional component, called the *head.* This component, placed at the beginning of the hypersequent, is used to store comparative plausibility formulas which are globally valid in a model. The comparative plausibility formulas are stored in the form of conditional blocks - the disjunction of comparative plausibility formulas introduced for the nested and hypersequent calculi. Since validity of comparative plausibility formulas does not change throughout the model, the rules allow blocks to “pass” between all components of the hypersequent.

Definition 4.9.1. Head-hypersequents have the following structure:

$$\Theta \mid\mid \Gamma_1 \Rightarrow \Delta_1 \mid \ \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

where $\Gamma_1 \ldots \Gamma_n$ and $\Delta_1 \ldots \Delta_n$ are multisets of formulas, and Θ is a multiset of conditional blocks. Again, a block $[A_1, \ldots, A_n \lhd C]$is a syntactic structure representing a disjunction of comparative plausibility formulas, interpreted as: $A_1 \preccurlyeq C \vee \cdots \vee A_n \preccurlyeq C$. Θ is called the *head* of the hypersequent; each $\Sigma_i \Rightarrow \Pi_i$ is called a *component* of the hypersequent.

The formula interpretation of hypersequents is given by:

$$([\Sigma_1 \lhd C_1], \ldots, [\Sigma_k \lhd C_k] \ \mid\mid \Gamma_1 \Rightarrow \Delta_1 \mid \ \ldots \mid \Gamma_n \Rightarrow \Delta_n)^{int} :=$$

$$\bigvee_{1 \leqslant i \leqslant k} \bigvee_{A \in \Sigma_i} (A \preccurlyeq C_i) \vee \Box(\bigwedge \Gamma_1 \to \bigvee \Delta_1) \vee \cdots \vee \Box(\bigwedge \Gamma_n \to \bigvee \Delta_n).$$

The components of the hypersequent are interpreted in the standard way in modal logics, each corresponding to a word of the model. Thus, for a hypersequent $\Theta \mid\mid \Gamma_1 \Rightarrow \Delta_1 \mid \ \ldots \mid \Gamma_n \Rightarrow \Delta_n$ to be valid in a model $\mathcal{M}$ at a world x, it must hold that either $x \Vdash A \preccurlyeq C$ for some $A \in \Sigma$ and some block $[\Sigma \lhd C]$ occurring in Θ, *or* $x \nVdash F$ for some F occurring in Γ_i, *or* $x \Vdash G$ for some G occurring in Δ_i, for $1 \leqslant i \leqslant n$.

The rules of the head-hypersequent calculi $\mathcal{HH}^i_{\mathsf{Strong^A}}$ are given in Figure 4.6. Initial sequents and propositional rules are the standard; the head component is not involved. Rules for comparative plausibility, com^A and jump^A display the same structure of the hypersequent rules defined for $\mathcal{SH}^i_{\mathbb{VTU}}$, with the difference that all the blocks are stored in the head - and can therefore be applied to all components. We have split the rule for total reflexivity into two rules, that differ for the rightmost premiss. Rule T_1 adds a formula to the same component; rule T_2 adds a formula to other components of the hypersequent. Rules W^A and C^A are defined similarly as the rules for the calculi $\mathcal{SH}^i_{\mathbb{VTU}}$. There is no need to add rules for absoluteness - the semantic condition is encompassed by the syntactic structure of the hypersequents.

Example 4.9.1. Derivation of Axiom (T): $(\bot \preccurlyeq \neg A) \to A$.

$$\dfrac{\dfrac{\dfrac{[\bot \lhd \bot] \mid\mid \overset{\triangledown}{\Rightarrow} A \mid \bot \Rightarrow \bot}{[\bot \lhd \bot] \mid\mid \Rightarrow A}\,\mathsf{jump}^A \qquad \dfrac{\mid\mid \bot \preccurlyeq \neg \overset{\triangledown}{A}, A \Rightarrow A}{\mid\mid \bot \preccurlyeq \neg A \Rightarrow A, \neg A}\,R\neg}{\mid\mid \bot \preccurlyeq \neg A \Rightarrow A}\,\mathsf{T}_1}{\mid\mid \Rightarrow (\bot \preccurlyeq \neg A) \to A}\,R\to$$

Derivation of axiom (A_1): $A \preccurlyeq B \to (\bot \preccurlyeq \neg(A \preccurlyeq B))$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[B, A \lhd B] \cdots \mid\mid A \preccurlyeq \overset{\triangledown}{B} \Rightarrow \mid \Rightarrow \bot \mid B \Rightarrow A, B}{[B, A \lhd B] \cdots \mid\mid A \preccurlyeq B \Rightarrow \mid \Rightarrow \bot}\,\mathsf{jump}^A \qquad \dfrac{[A \lhd A] \cdots \mid\mid A \preccurlyeq \overset{\triangledown}{B} \Rightarrow \mid \Rightarrow \bot \mid A \Rightarrow A}{[A \lhd A] \cdots \mid\mid A \preccurlyeq B \Rightarrow \mid \Rightarrow \bot}\,\mathsf{jump}^A}{[A \lhd B], [\bot \lhd \neg(A \preccurlyeq B)] \mid\mid A \preccurlyeq B \Rightarrow \mid \Rightarrow \bot}\,\mathsf{L} \preccurlyeq}{[\bot \lhd \neg(A \preccurlyeq B)] \mid\mid A \preccurlyeq B \Rightarrow \mid \Rightarrow A \preccurlyeq B, \bot}\,\mathsf{R} \preccurlyeq}{[\bot \lhd \neg(A \preccurlyeq B)] \mid\mid A \preccurlyeq B \Rightarrow \mid \neg(A \preccurlyeq B) \Rightarrow \bot}\,\mathsf{R}\neg}{[\bot \lhd \neg(A \preccurlyeq B)] \mid\mid A \preccurlyeq B \Rightarrow}\,\mathsf{jump}^A}{\mid\mid A \preccurlyeq B \Rightarrow \bot \preccurlyeq \neg(A \preccurlyeq B)}\,\mathsf{R} \preccurlyeq^A}{\mid\mid \Rightarrow A \preccurlyeq B \to (\bot \preccurlyeq \neg(A \preccurlyeq B))}\,\mathsf{R} \to$$

Initial sequents

$$\frac{}{\Theta \,||\, p, \Sigma \Rightarrow \Pi, p \,|\, \mathcal{G}}\ \mathsf{init} \qquad \frac{}{\Theta \,||\, \bot \Sigma \Rightarrow \Pi \,|\, \mathcal{G}}\ \bot_{\mathsf{L}}$$

Propositional rules

$$\frac{\Theta \,||\, B, \Gamma \Rightarrow \Delta \,|\, \mathcal{G} \quad \Theta \,||\, \Gamma \Rightarrow \Delta, A \,|\, \mathcal{G}}{\Theta \,||\, A \to B, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}}\ \mathsf{L} \to \qquad \frac{\Theta \,||\, A, \Gamma \Rightarrow \Delta, B \,|\, \mathcal{G}}{\Theta \,||\, \Gamma \Rightarrow \Delta, A \to B \,|\, \mathcal{G}}\ \mathsf{R} \to$$

Conditional rules

$$\frac{\Theta, [A \lhd B] \,||\, \Sigma \Rightarrow \Pi \,|\, \mathcal{G}}{\Theta \,||\, \Sigma \Rightarrow \Pi, A \preccurlyeq B \,|\, \mathcal{G}}\ \mathsf{R} \preccurlyeq^A \qquad \frac{\Theta, [\Sigma \lhd C] \,||\, \mathcal{G} \,|\, C \Rightarrow \Sigma}{\Theta, [\Sigma \lhd C] \,||\, \mathcal{G}}\ \mathsf{jump}^A$$

$$\frac{\Theta, [B, \Sigma \lhd C] \,||\, A \preccurlyeq B, \Sigma \Rightarrow \Pi \,|\, \mathcal{G} \quad \Theta, [\Sigma \lhd C], [\Sigma \lhd A] \,||\, A \preccurlyeq B, \Sigma \Rightarrow \Pi \,|\, \mathcal{G}}{\Theta, [\Sigma \lhd C] \,||\, A \preccurlyeq B, \Sigma \Rightarrow \Pi \,|\, \mathcal{G}}\ \mathsf{L} \preccurlyeq^A$$

$$\frac{\Theta, [\Sigma, \Pi \lhd A], [\Pi \lhd B] \,||\, \mathcal{G} \quad \Theta, [\Sigma \lhd A], [\Sigma, \Pi \lhd B] \,||\, \mathcal{G}}{\Theta, [\Sigma \lhd A], [\Pi \lhd B] \,||\, \mathcal{G}}\ \mathsf{com}^A$$

Rules for reflexivity

$$\frac{\Theta, [\bot \lhd A] \,||\, A \preccurlyeq B, \Gamma \Rightarrow \Delta \,|\mathcal{G} \quad \Theta \,||\, A \preccurlyeq B, \Gamma \Rightarrow \Delta, B \,|\mathcal{G}}{\Theta \,||\, A \preccurlyeq B, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}}\ \mathsf{T}_1$$

$$\frac{\Theta, [\bot \lhd A] \,||\, A \preccurlyeq B, \Gamma_1 \Rightarrow \Delta_1 \,|\Gamma_2 \Rightarrow \Delta_2 \,|\mathcal{G} \quad \Theta \,||\, A \preccurlyeq B, \Gamma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2, B \,|\, \mathcal{G}}{\Theta \,||\, A \preccurlyeq B, \Gamma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2 \,|\mathcal{G}}\ \mathsf{T}_2$$

Rules for extensions

$$\frac{[\Sigma \lhd A], \Theta \,||\, \Gamma \Rightarrow \Delta, \Sigma \,|\, \mathcal{G}}{[\Sigma \lhd A], \Theta \,||\, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}}\ \mathsf{W}^A \qquad \frac{[\Sigma \lhd C] \,||\, C, \Gamma \Rightarrow \Delta, \Sigma}{[\Sigma \lhd C] \,||\, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}}\ \mathsf{jump_C}^A$$

$$\frac{\Theta \,||\, A, A \preccurlyeq B, \Gamma \Rightarrow \Delta \,|\, \mathcal{G} \quad \Theta \,||\, A \preccurlyeq B, \Gamma \Rightarrow \Delta, B \,|\, \mathcal{G}}{\Theta \,||\, A \preccurlyeq B, \Gamma \Rightarrow \Delta \,|\, \mathcal{G}}\ \mathsf{C}^A$$

$$\mathcal{HH}^i_{\mathbb{VTA}} = \{\mathsf{init}, \bot, \mathsf{R} \to, \mathsf{L} \to, \mathsf{R} \preccurlyeq^A, \mathsf{L} \preccurlyeq^A, \mathsf{jump}^A, \mathsf{com}^A, \mathsf{T}_1, \mathsf{T}_2\}$$

$$\mathcal{HH}^i_{\mathbb{VWA}} = \mathcal{HH}^i_{\mathbb{VTA}} \cup \{\mathsf{W}^A\} \qquad \mathcal{HH}^i_{\mathbb{VCA}} = \mathcal{HH}^i_{\mathbb{VWA}} \cup \{\mathsf{C}^A, \mathsf{jump_C}^A\}$$

Figure 4.6: Hypersequent calculus $\mathcal{H}_{\mathbb{VTA}}$ and extensions

Theorem 4.9.1 (Soundness)**.** If a head-hypersequent $\mathcal{G}$ is derivable in $\mathcal{HH}^i_{\mathsf{Strong^A}}$ then $(\mathcal{G})^{int}$ is a theorem of the corresponding logic.

Proof. By inductive on the height of the derivation of the premisses, assuming that the premisses are valid in all sphere models for $\mathbb{VTA}$ (resp. $\mathbb{VWA}$ and $\mathbb{VCA}$) while the conclusion is not. We show the case of rules T_1; the cases for the other rules are similar to the proof of soundness for $\mathcal{SH}^i_{\mathsf{Strong}}$.
Rule T_1: suppose the conclusion of the rule is false at model $\mathcal{M}$ and world x. Thus, we have:

1. for all $T \in \Theta$, $x \nVdash T$;
2. $x \Vdash \Box(A \preccurlyeq B)$, that is for all $y \in \bigcup N(x)$ it holds that for all $\alpha \in N(y)$ if $\alpha \Vdash^{\exists} B$ then $\alpha \Vdash^{\exists} A$;
3. $x \Vdash \Box \bigwedge \Gamma$;
4. $x \nVdash \Box \bigvee \Delta$;
5. $x \nVdash \iota(\mathcal{G})$.

Since the premisses are valid, it holds that:

6. $x \Vdash \bot \preccurlyeq A$ (from validity of the left premiss, 1, 2, 3, 4 and 5); thus, for all $\beta \in N(x)$ if $\beta \Vdash^{\exists} A$ then $\beta \Vdash^{\exists} \bot$;
7. $x \Vdash \Box B$, that is for all $y \in \bigcup N(x)$ it holds that $y \Vdash B$ (from validity of the second premiss, 1, 2, 3, 4 and 5).

By total reflexivity we have that there exists $\alpha \in N(x)$ such that $x \in \alpha$. Since $x \in \bigcup N(x)$, from 7 we have $x \Vdash B$; thus, $\alpha \Vdash^{\exists} B$. By 2 and absoluteness we have $\alpha \Vdash^{\exists} A$. By 5 and absoluteness, we have $\alpha \Vdash^{\exists} \bot$, ending up in a contradiction.

Proof of T_2 proceeds in exactly the same way, since validity of the right-hand premiss yields $x \Vdash \Box B$. $\square$

Similarly to hypersequent calculi $\mathcal{SH}^{i}_{\mathsf{Strong}}$, weakening and contraction are admissible in $\mathcal{HH}^{i}_{\mathsf{Strong^A}}$. We omit the proofs of the following lemmas, similar to the proofs of Lemmas 4.7.2, 4.7.3 and 4.7.4.

Lemma 4.9.2. The structural rules of weakening on formulas and on blocks are height-preserving admissible in $\mathcal{HH}^{i}_{\mathsf{Strong^A}}$.

$$\frac{\Theta \mid\mid \mathcal{G} \mid \Gamma \Rightarrow \Delta}{\Theta \mid\mid \Sigma, \Gamma \Rightarrow \Delta, \Pi \mid \mathcal{G}}\ \mathsf{Wk} \qquad \frac{\Theta \mid\mid \mathcal{G}}{\Theta \mid\mid \Rightarrow \mid \mathcal{G}}\ \mathsf{Wk_E}$$

$$\frac{\Theta \mid\mid \Gamma \Rightarrow \Delta \mid \mathcal{G}}{[\Sigma \lhd C], \Theta \mid\mid \Gamma \Rightarrow \Delta \mid \mathcal{G}}\ \mathsf{Wk_i} \qquad \frac{[\Sigma \lhd C], \Theta \mid\mid \Gamma \Rightarrow \Delta, x \mid \mathcal{G}}{[\Sigma, \Omega \lhd C], \Theta \mid\mid \Gamma \Rightarrow \Delta, \mid \mathcal{G}}\ \mathsf{Wk_B}$$

Lemma 4.9.3. All the rules of $\mathcal{H}_{\mathbb{VTA}}$, $\mathcal{HH}^{i}_{\mathsf{Strong^A}}$ are height-preserving invertible.

Lemma 4.9.4. The structural rules of contraction on formulas and on blocks are height-preserving admissible in $\mathcal{HH}^{i}_{\mathsf{Strong^A}}$.

$$\frac{\Theta \mid\mid A, A, \Gamma \Rightarrow \Delta \mid \mathcal{G}}{\Theta \mid\mid A, \Gamma \Rightarrow \Delta \mid \mathcal{G}}\ \mathsf{Ctr_L} \quad \frac{\Theta \mid\mid \Gamma \Rightarrow \Delta, A, A \mid \mathcal{G}}{\Theta \mid\mid \Gamma \Rightarrow \Delta, A \mid \mathcal{G}}\ \mathsf{Ctr_R}$$

$$\frac{[\Sigma \lhd A], [\Sigma \lhd A], \Theta \mid\mid \Gamma \Rightarrow \Delta \mid \mathcal{G}}{[\Sigma \lhd A], \Theta \mid\mid \Gamma \Rightarrow \Delta \mid \mathcal{G}}\ \mathsf{Ctr_B} \quad \frac{[\Sigma, A, A \lhd B], \Theta \mid\mid \Gamma \Rightarrow \Delta \mid \mathcal{G}}{[\Sigma, A \lhd B], \Theta \mid\mid \Gamma \Rightarrow \Delta \mid \mathcal{G}}\ \mathsf{Ctr_i}$$

4.10 Semantic completeness of $\mathcal{HH}^{i}_{\mathsf{Strong^A}}$

To prove completeness of $\mathcal{HH}^{i}_{\mathsf{Strong^A}}$ we introduce (as usual) cumulative versions of the calculi, which keep all formulas and blocks in the premisses. Then, the proof proceeds by constructing a countermodel from the upper head-hypersequent of a failed derivation branch. The rules which need to be modified in their cumulative versions are the following:

$$\frac{\Theta \mid\mid A \to B, B, \Gamma \Rightarrow \Delta \mid \mathcal{G} \quad \Theta \mid\mid A \to B, \Gamma \Rightarrow \Delta, A \mid \mathcal{G}}{\Theta \mid\mid A \to B, \Gamma \Rightarrow \Delta \mid \mathcal{G}}\ \mathsf{L} \to^{c}$$

$$\frac{\Theta \mid\mid A, \Gamma \Rightarrow \Delta, A \to B, B \mid \mathcal{G}}{\Theta \mid\mid \Gamma \Rightarrow \Delta, A \to B \mid \mathcal{G}}\ \mathsf{R} \to^{c} \quad \frac{\Theta, [A \lhd B] \mid\mid \Gamma \Rightarrow \Delta, A \preccurlyeq B \mid \mathcal{G}}{\Theta \mid\mid \Gamma \Rightarrow \Delta, A \preccurlyeq B \mid \mathcal{G}}\ \mathsf{R} \preccurlyeq^{A}_{c}$$

$$\frac{\Theta, [\Sigma, \Pi \lhd A], [\Sigma \lhd A], [\Pi \lhd B] \mid\mid \mathcal{G} \quad \Theta, [\Sigma \lhd A], [\Sigma, \Pi \lhd B], [\Pi \lhd B] \mid\mid \mathcal{G}}{\Theta, [\Sigma \lhd A], [\Pi \lhd B] \mid\mid \mathcal{G}}\ \mathsf{com}^{A}_{c}$$

Definition 4.10.1. Let $\mathcal{H}$ be a head-hypersequent of the form $\Theta \,||\mathcal{G}$. $\mathcal{H}$ is *saturated* with respect to $\mathcal{H}^{c}_{\mathbb{VTA}}$ if it not an instance of init or $\bot_{\mathsf{L}}$ and it satisfies the following conditions:

1. $(\mathsf{L} \rightarrow^{c})$ if $A \rightarrow B, \Gamma \Rightarrow \Delta \in \mathcal{G}$, then $B \in \Gamma$ or $A \in \Delta$;
2. $(\mathsf{R} \rightarrow^{c})$ if $\Gamma \Rightarrow \Delta, A \rightarrow B \in \mathcal{G}$, then $A \in \Gamma$ and $B \in \Delta$;
3. $(\mathsf{R} \preccurlyeq^{A}_{c})$ if $\Gamma \Rightarrow A \preccurlyeq B \in \mathcal{G}$, then $[\Sigma, A \lhd B] \in \Theta$, for some Σ;
4. $(\mathsf{L} \preccurlyeq^{A})$ if $[\Sigma \lhd C] \in \Theta$ and $A \preccurlyeq B, \Gamma \Rightarrow \Delta \in \mathcal{G}$, then either $B \in \Sigma$ or $[\Sigma \lhd A] \in \Theta$;
5. (com^{A}_{c}) if $[\Sigma \lhd A]\,, [\Sigma \lhd B] \in \Theta$, then $\Sigma \subseteq \Pi$ or $\Pi \subseteq \Sigma$;
6. (jump^{A}) if $[\Sigma \lhd C] \in \Theta$, then $C, \Lambda \Rightarrow \Omega, \Sigma \in \mathcal{G}$;
7. (T_1) if $A \preccurlyeq B, \Gamma \Rightarrow \Delta \in \mathcal{G}$, then either $[\bot \lhd A] \in \Theta$ or $B \in \Delta$;
8. (T_2) if $A \preccurlyeq B, \Gamma \Rightarrow \Delta \in \mathcal{G}$, then either $[\bot \lhd A] \in \Theta$ or for all $\Lambda \Rightarrow \Omega$ components of $\mathcal{G}$, $B \in \Omega$;
9. (jumpc^{A}) if $[\Sigma \lhd A] \in \Theta$ and $\Gamma \Rightarrow \Delta \in \mathcal{G}$, then $A \in \Gamma$ and $\Sigma \subseteq \Delta$.

Furthermore, $\mathcal{H}$ is $\mathcal{HH}^{i}_{\mathbb{VWA}}$ saturated (respectively: $\mathcal{HH}^{i}_{\mathbb{VWA}}$) if the following conditions hold.

1. (W^{A}) if $[\Sigma \lhd A] \in \Theta$, $\Gamma \Rightarrow \Delta \in \mathcal{G}$, then $\Sigma \subseteq \Delta$;
2. (C^{A}) if $\Gamma, C \preccurlyeq D \Rightarrow \Delta \in \mathcal{G}$, then $C \in \Gamma$ or $D \in \Delta$.

Let $\mathcal{H}$ be a saturated head-hypersequent of the form $\Theta \,||\, \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$. We construct a countermodel $\mathcal{M}_{\mathcal{H}}$ for it similarly as done in Section 4.8:

- $W := \{1, \ldots, n\}$;
- $[\![p]\!] := \{i \leqslant n : p \in \Gamma_i\}$.

We introduce a world for each component of the hypersequent, satisfying atomic formulas occurring in the antecedent of the component. To account for the condition of absoluteness, we set the system of neighbourhood to be the same for all the worlds. Thus, for each $i \leqslant n$, we define the same system of neighbourhood $S(i)$ from the blocks occurring in the head of the hypersequent:

$$\Theta = [\Sigma_1 \lhd C_1]\,, \ldots, [\Sigma_n \lhd C_n]$$

By the saturation condition associated to (com^{A}_{c}), we have that the antecedents of the blocks are ordered as follows: $\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots \subseteq \Sigma_n$. By the saturation condition associated to jump^{A}, for every $j \leqslant k$ there is a component m_j such that $\Lambda_{m_j} \Rightarrow \Omega_{m_j} \in \mathcal{G}$, and $A_j \in \Lambda_{m_j}$ and $\Sigma_j \subseteq \Omega_{m_j}$. Thus, for all $i \in W$, set:

$$N(i) := \{\{m_k\}, \{m_k, m_{k-1}\}, \ldots, \{m_k, \ldots, m_1\}, W\}$$

Lemma 4.10.1. If $\mathcal{H}$ is a head-hypersequent saturated with respect to $\mathcal{H}_{\mathbb{VTA}}$ the structure $\mathcal{M}_{\mathcal{H}}$ is a sphere model for $\mathcal{H}_{\mathbb{VTA}}$.

Proof. Nesting of neighbourhoods holds in $\mathcal{M}_{\mathcal{H}}$, since $\{m_k\} \subseteq \{m_k, m_{k-1}\} \subseteq \cdots \subseteq \{m_k, \ldots, m_1\} \subseteq W$. Reflexivity holds, since $W \in N(i)$, for all $i \in W$. Absoluteness holds, since for every $v, w \in W$ it holds that $N(v) = N(w)$. $\square$

Lemma 4.10.2. Let $\mathcal{H}$ be a saturated $\mathcal{H}_{\mathbb{VTA}}$ head-hypersequent, as defined in 4.10.1, and let $\mathcal{M}_{\mathcal{H}}$ be a $\mathbb{VTA}$ sphere model. For all $i \leqslant n$ world of the countermodel, corresponding to a component $\Gamma_i \Rightarrow \Delta_i \in \mathcal{G}$, we show that:

1. for all $A \in \Gamma_i$, $\mathcal{M}_{\mathcal{H}}, i \Vdash A$;
2. for all $A \in \Delta_i$, $\mathcal{M}_{\mathcal{H}}, i \nVdash A$;
3. for all $[\Sigma \lhd C] \in \Theta$, $\mathcal{M}_{\mathcal{H}}, i \nVdash \bigvee_{A \in \Sigma}(A \preccurlyeq C)$.

Proof. Cases 1 and 2 are proved by mutual inductive on the complexity of the formula A[8]. Propositional cases are immediate; we consider only case $A \equiv E \preccurlyeq F$. Proof of 3 uses 2.

1. Suppose $E \preccurlyeq F \in \Gamma_i$. For all $\alpha \in S$, we have to show that either $\alpha \nVdash^{\exists} F$ or $\alpha \Vdash^{\exists} E$. In case $\alpha \neq W$ we have $\alpha = \{m_k, \ldots, m_j\}$, for some $j \leqslant k$. Each $m_l \in \alpha$ comes from a component $C_l, \Lambda_l \Rightarrow \Omega_l, \Sigma_l$ generated from a block $[\Sigma_l \lhd C_l] \in \Theta$. Let us consider m_j, and its associated component $C_j, \Lambda_j \Rightarrow \Omega_j, \Sigma_j$. For the saturation condition associated to H_L, either $F \in \Gamma_j$ or $E = C_j$. In the former case we have that $\Sigma_j \subseteq \cdots \subseteq \Sigma_k$; by inductive hypothesis $m_i \nvDash F$, for $i = \{j, \ldots, k\}$. Thus, $\alpha \Vdash^{\forall} \neg F$. Otherwise, if $E = C_j$, let us consider the component $C_j, \Lambda_j \Rightarrow \Omega_j, \Sigma_j$. By inductive hypothesis, $\mathcal{M}_{\mathcal{H}}, m_j \Vdash^{\exists} E$; thus, $\alpha \Vdash^{\exists} E$.
In case $\alpha = W$, by the saturation condition associated to T_1 we have that either $[\bot \lhd E] \in \Theta$ or $F \in P_i$ for all components $\Sigma_i \Rightarrow \Pi_i \in \mathcal{G}$. In the former case, by the saturation condition associated to jump there will be a component $E, \Lambda_j \Rightarrow \Omega_j, \bot$; thus, by inductive hypothesis $m_j \Vdash^{\exists} E$, and $W \Vdash^{\exists} E$. In the latter case, we get by inductive hypothesis that for all $m_i \in W$, $m_j \nVdash^{\exists} F$; thus,$W \Vdash^{\forall} \neg F$.
2. Suppose $E \preccurlyeq F \in \Delta_i$. For some $\alpha \in N(i)$, we have to show that $\alpha \Vdash^{\exists} F$ and $\alpha \nVdash^{\exists} E$. By saturation condition associated to $\mathsf{R}\preccurlyeq^{A}$, there is a $j \leqslant k$ such that $C_j = F$ and $E \in \Gamma_j$. Consider the world m_j associated to the component $C_j, \Lambda_j \Rightarrow \Omega_j, \Sigma_j$. By inductive hypothesis it holds that $m_j \Vdash^{\exists} F$; moreover, since $\Sigma_j \subseteq \Sigma_j + 1 \subseteq \cdots \subseteq \Sigma_k$, it holds by inductive hypothesis that $m_i \nVdash^{\exists} E$ for $i = \{j, \ldots, k\}$. Thus for $\alpha = \{m_j, \ldots, m_k\}$ we get that $\alpha \Vdash^{\exists} F$ and $\alpha \nVdash^{\exists} E$. $\square$

To extend the model construction to $\mathbb{VWA}$, let us consider how models for this logic are structured. For the condition of weak centering, every world of the model must be included into *all* the systems of spheres associated to that world. Along with the condition of absoluteness, this means that there is just one sphere, containing all the worlds of the model. Thus, given a head-hypersequent $\mathcal{H}$ we define $\mathcal{M}^{w}_{\mathcal{H}}$ with W and $[\![\ ]\!]$ as in $\mathcal{M}_{\mathcal{H}}$, and for all worlds $i \in W$, let

$$N(i) := \{\{m_k, 1, \ldots, n\}, \{m_k, m_{k-1}, 1, \ldots, n\}, \ldots, \{m_k, \ldots, m_1, 1, \ldots, n\}, W\}.$$

The condition can be equivalently stated as $N(i) := \{W\}$. It can be immediately verified that such structures are sphere models for $\mathbb{VWA}$. Then, it is possible to extend Lemma 4.10.4 as follows.

Lemma 4.10.3. Let $\mathcal{H}^w$ be a saturated $\mathcal{HH}^{i}_{\mathbb{VWA}}$ hypersequent as defined above, and $\mathcal{M}^{w}_{\mathcal{H}}$ a $\mathbb{VWA}$ sphere model. For $i \in W$ and $1 \leqslant k \leqslant n$ we show that:

1. for all $A \in \Gamma_k$, $\mathcal{M}^{w}_{\mathcal{H}}, i \Vdash A$;
2. for all $A \in \Delta_k$, $\mathcal{M}^{w}_{\mathcal{H}}, i \nVdash A$;
3. for all $[\Sigma \lhd C] \in \Theta$, $\mathcal{M}^{W}_{\mathcal{H}}, i \nVdash \bigvee_{A \in \Sigma}(A \preccurlyeq C)$.

Proof. The proof is by inductive on the complexity of formulas. We show only cases 1 and 2, considering $A = E \preccurlyeq F$. Proof of 3 makes use of 1 and 2.

1. Suppose $E \preccurlyeq F \in \Gamma_i$, for some $1 \leqslant i \leqslant n$. We have to prove that for all $\alpha \in S$, either $\alpha \nVdash^{\exists} F$ or $\alpha \Vdash^{\exists} E$. In this case, however, $\alpha = W$. By the saturation conditions associated to T_1 we have that either $[\bot \lhd E]$ belongs to Θ or F belongs to Δ_i. In the former case, by the saturation condition associated to jump^{A} we have that E occurs in Γ_k, for some $1 \leqslant k \leqslant n$ and $i \neq k$. However, $k \in \alpha$, and by inductive hypothesis $\alpha \Vdash^{\exists} E$. Otherwise, from F belongs to Δ_i and inductive hypothesis we obtain $\alpha \nVdash^{\exists} F$.

[8]Refer to Definition 4.8.2 for the notion of complexity of a formula.

2. Suppose $E \preccurlyeq F \in \Delta_i$, for some $1 \leqslant i \leqslant n$. We have to show that for $\alpha = W$ it holds that $\alpha \Vdash^{\exists} F$ and $\alpha \nVdash^{\exists} E$. By the saturation conditions associated to $\mathsf{R}\preccurlyeq^{A}$ and jump^{A}, we have that there exists a k such that $F \in \Gamma_k$ and $E \in \Delta_k$. Thus, by inductive hypothesis, $\alpha \Vdash^{\exists} F$ and $\alpha \nVdash^{\exists} E$. □

To prove completeness for $\mathbb{VCA}$ the model can be further simplified, since models for this logic are composed of just one world and just one sphere around that world. Moreover, due to the presence of the rule jump_A, a head-hypersequent in $\mathcal{HH}^i_{\mathbb{VCA}}$ has the form $\Theta \,||\, \Gamma \Rightarrow \Delta$, with just one component. Let $\mathcal{H}$ be such a hypersequent. We construct a countermodel $\mathcal{M}^c_{\mathcal{H}}$ as follows:

- $W = \{i\}$;
- $N(i) = \{\{i\}\}$, for all $i \in W$;
- $[\![p]\!] := \{i \leqslant n : p \in \Gamma_k\}$ for $1 \leqslant k \leqslant n$.

Model $\mathcal{M}^c_{\mathcal{H}}$ trivially satisfies all conditions of a sphere model for $\mathbb{VCA}$: nesting, non-emptiness, centering and absoluteness. The proof of Lemma 4.10.2 is modified as follows.

Lemma 4.10.4. Let $\mathcal{H}^c$ be a saturated $\mathcal{HH}^i_{\mathbb{VCA}}$ hypersequent as defined above, and $\mathcal{M}^c_{\mathcal{H}}$ a $\mathbb{VTA}$ neighbourhood model. For $i \in W$ and $1 \leqslant k \leqslant n$ we show that:

1. for all $A \in \Gamma_k$, $\mathcal{M}^c_{\mathcal{H}}, i \Vdash A$;
2. for all $A \in \Delta_k$, $\mathcal{M}^c_{\mathcal{H}}, i \nVdash A$;
3. for all $[\Sigma \lhd C] \in \Theta$, $\mathcal{M}^c_{\mathcal{H}}, i \nVdash \bigvee_{A \in \Sigma}(A \preccurlyeq C)$.

Proof. The proof is by inductive on the complexity of formulas. Again, we show only cases 1 and 2, subcase $A = E \preccurlyeq F$.

1. $E \preccurlyeq F \in \Gamma_k$. There is just one non-empty sphere in the model, $\alpha = \{i\}$. We have to show that either $\alpha \nVdash^{\exists} F$ or $\alpha \Vdash^{\exists} E$. By the saturation condition associated to C^A, we have that either $E \in \Gamma_k$ or $F \in \Delta_k$. By inductive hypothesis, this means that either $i \Vdash E$, from which we have $\alpha \Vdash^{\exists} E$, or $i \nVdash F$, thus $\alpha \nVdash^{\exists} F$.

2. $E \preccurlyeq F \in \Delta_k$. We have to show that for the only sphere in the model, α, it holds that $\alpha \Vdash^{\exists} F$ and $\alpha \nVdash^{\exists} E$. By the saturation conditions associated to $\mathsf{R}\preccurlyeq^{A}_{c}$ and jump^{A} we have that $F \in \Gamma_k$ and $E \in \Delta_k$. By inductive hypothesis $i \vDash F$ and $i \nVdash E$; thus, $\alpha \Vdash^{\exists} F$ and $\alpha \nVdash^{\exists} E$. □

From Lemma 4.10.2 and Lemma 4.10.4 we obtain:

Lemma 4.10.5. For $\mathsf{Strong^A} \in \{\mathbb{VTA}, \mathbb{VWA}, \mathbb{VCA}\}$, let $\mathcal{H} = \Theta \,||\, \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ be a $\mathsf{Strong^A}$-saturated head-hypersequent. Let $\mathcal{M}_{\mathcal{H}}$ be a model for $\mathsf{Strong^A}$, as defined above. It holds that:

- for any $i \in W$ associated to sequent $\Gamma_i \Rightarrow \Delta_i$, $\mathcal{M}_{\mathcal{H}}, i \nVdash (\Gamma_i \Rightarrow \Delta_i)^{int}$;
- for any $i \in W$, $[\Sigma \lhd C]$it holds that $\mathcal{M}_{\mathcal{H}}, i \nVdash ([\Sigma \lhd C])^{int}$;
- for any $i \in W$, it holds that $\mathcal{M}_{\mathcal{H}}, i \nVdash (\mathcal{H})^{int}$

To ensure termination we define, as for $\mathcal{SH}^i_{\mathsf{Strong}}$, the notion of local loop checking to obtain backwards termination of the calculi. We set that rules are not applied if there is a premiss from which the conclusion is derivable using structural rules. Then, from admissibility of weakening and contraction we can assume that our derivations consists only of duplication-free sequents. Since all the rules have the subformula property, the number of duplication-free sequents is bounded in the number of subformulas of that sequent. Thus, we have the following:

Proposition 4.10.6. Adding a local loop checking strategy, backwards proof search in $\mathcal{SH}^i_{\mathsf{Strong}}$ terminates, and every leaf of the resulting proof tree is either an axiom or a saturated head-hypersequent.

Theorem 4.10.7 (Completeness)**.** If $(\mathcal{H})^{int}$ is a theorem of $\mathsf{Strong}^{\mathsf{A}}$, then $\mathcal{HH}^i_{\mathsf{Strong}^{\mathsf{A}}} \vdash \mathcal{H}$.

Proof. By the previous proposition, backwards proof search with root $\mathcal{H}$ terminates, with every leaf either instance of init or $\bot_{\mathsf{L}}$ or a saturated head-hypersequent. However, a leaf cannot be a saturated head-hypersequent: if it were so, we could be a countermodel for it that, by monotonicity, would falsify also $\mathcal{G}$, against the hypothesis. $\square$

As done for calculi $\mathcal{SH}^i_{\mathsf{Strong}}$, we can define an immediate (non-optimal) procedure to decide validity of a formula in $\mathsf{Strong}^{\mathsf{A}}$: to check whether a formula A is derivable, it suffices to choose a duplication-free head-hypersequent containing only subformulas of A and containing a component $\Gamma \Rightarrow \Delta, A$. If this is not possible, backwards proof search yields a proof of $\Rightarrow A$; otherwise, the head-hypersequent gives rise to a countermodel. The size of the duplication-free head-hypersequent is bounded exponentially by the number of subformulas of A; thus, the complexity of our decision procedure would be co-NEXPTIME. This bound is far from optimal since, according to [28], conditional logics featuring absoluteness are NP-complete.

4.11 To sum up

In this chapter we have presented three families of internal calculi for Lewis' family of conditional logics: nested calculi $\mathcal{I}^{\mathsf{i}}_{\mathsf{Base}}$, for $\mathbb{V}$ and some of its extensions, extended hypersequent calculi $\mathcal{SH}^i_{\mathsf{Strong}}$, for $\mathbb{VTU}$ and its extensions, and the head-hypersequent systems $\mathcal{HH}^i_{\mathsf{Strong}^{\mathsf{A}}}$, for $\mathbb{VTA}$ and its extensions. In order to capture conditional logics, more expressive than modal logics, it was necessary to extend the structure of sequents. More precisely, to capture logics whose outer modal logic is weaker than **S5** we have employed a kind of nested sequent calculus, while to capture systems whose outer modal logic is **S5** we used hypersequents.

The attentive reader might have noticed that there are four logics for which we have not given a full treatment. These are $\mathbb{VU}$ and $\mathbb{VNU}$, for which we have not given any proof system, and $\mathbb{VA}$ and $\mathbb{VNA}$, for which we have given a nested sequent calculus but no direct proof of completeness. The outer modal logics for these systems are **K45** ($\mathbb{VU}$ and $\mathbb{VA}$) and **D45** ($\mathbb{VNU}$ and $\mathbb{VNA}$). It would be possible to give proof systems for these logics in terms of *grafted hypersequents*. This formalism was introduced by Lellmann and Kuznets in [51], as a proof system for **K5** and **K45**. In these systems there is a difference between the actual world - in which reflexivity does *not* hold - and the worlds "seen" from the actual world - in which reflexivity holds. The framework of grafted hypersequents is something in between nested and hypersequent calculi, and it can be particularly useful to treat weak systems of modal logics. Grafted hypersequents are composed of a head (the "trunk") which is a nested sequent of bounded depth, and a body (the "crown") which consists of a hypersequent. In our case, the trunk would consist of a nested sequent of depth at most one, corresponding to the formulas relative to the actual world. Each component of the hypersequent would correspond to the formulas relative to a world "seen" from the actual world. Then, by defining two sets of rules, one for the trunk and one for the crown, it would be possible to differentiate between the actual world and the worlds seen from it. The definition of such calculi for $\mathbb{VU}$, $\mathbb{VA}$, $\mathbb{VNU}$ and $\mathbb{VNA}$ is object of current research.

Chapter 5

Relations between proof systems

"What about the reality where Hitler cured cancer, Morty?
The answer is: Don't think about it."
– Rick, *Rick and Morty*

This chapter presents the two directions of a mapping between two proof systems for conditional logic $\mathbb{V}$: labelled calculus **G3V** and internal calculus $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$. Both proof systems were introduced in the previous chapters. The interest of the result lies in the fact that the two calculi are *prima facie* quite different - one displays labels in correspondence with semantic elements of the models, while the other employs blocks, syntactic structures corresponding to disjunctions of comparative plausibility formulas. The present mapping can be inscribed into the more general research on the interrelations between proof systems for modal logics.

5.1 A case study

In the previous chapters we have presented different standard proof systems for conditional logics: labelled calculi for $\mathbb{PCL}$ and its extensions (Chapter 3), and internal calculi for $\mathbb{V}$ and its extensions (Chapter 4). The calculi have different properties: for instance, the proof of cut admissibility for labelled systems is almost straightforward, due to invertibility of the rules; but these systems require a quite complex proof strategy to ensure termination. Conversely, of all the internal calculi presented in Chapter 4 we were able to prove cut-elimination only for one system, $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$, and by means of a quite complex proof. Proving termination for internal calculi is much easier, and derivations with such calculi tend to be much shorter than derivations using labelled systems.

Moreover, rules of labelled sequent systems have an immediate explanation in terms of the semantics for the corresponding logics: the labels of these calculi are syntactic correspondents of elements (worlds or neighbourhood) of the models. In comparison, rules of internal systems are less intuitive to understand: the relation with elements of the model is hidden in the structure of sequents. In Chapter 4 we have offered some intuitive explanation relating blocks to spheres; in this chapter we motivate it by means of formal arguments.

The question whether it is possible to establish a formal correspondence between labelled and internal proof systems for conditional logics arises as a natural one. Logic $\mathbb{V}$ has been chosen as a case study, since it is the basic system of Lewis' family and the weakest conditional logic having both an internal and a labelled proof system. The labelled sequent calculus is **G3V**, exposed in Chapter 3; the internal calculus is $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$, from Chapter 4.

Generally speaking, there are several advantages in establishing mappings between different proof systems. On one side, once a translation is defined and proved to be sound, meta-theoretical results proved from one calculus can be transferred into the other. On the other side, establishing the two directions of a mapping helps to give a full understanding of both systems. As for modal logics, several results on the relationships between different proof systems can be found in the literature. Refer to [89, pp. 116, 206] for a survey, along with a conjecture of a general interpretability of tree hypersequents into labelled calculi. This conjecture was confirmed by Goré and Ramanayake in [40] through an embedding of nested sequent calculi and tree-hypersequent calculi into a suitable subclass of labelled calculi, those in which the relational structure forms a tree. A similar result has been proved for nested sequent calculi and labelled tableaux by Fitting in [27]. More recently, Pimentel developed translations between labelled and nested sequent calculi for normal and non-normal systems of modal logics [86] .

As happens with equivalence results between labelled and internal calculi for modal logics in [40], in our case study the translation from the internal calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ to the labelled system **G3V** can be directly specified by adding to the labelled calculus a few admissible rules. The direction from labelled to internal is considerably more difficult. The problem is that **G3V** derivable sequents are *more* than $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivable sequents. Thus, in a **G3V** derivation not all sequents can be translated into the language of $\mathbb{V}$. This overload has to be disciplined, and the core of the procedure lies in the identification of a "normal form" for **G3V** derivations, basically corresponding to the requirement that the relational structure of a derivation forms a tree, and in the Jump lemma (Lemma 5.4.2), which allows to focus om a subset of formulas of a labelled sequent - a derivable subset that can be translated. We show how, thanks to these restrictions, we are able to translate **G3V** derivations into $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivations.

For both directions of the mapping we define a function taking care of translating sequents (denoted with t, s) and a function taking care of translating full derivations (denoted with $\{\ \}$, $[\![\ ]\!]$). Soundness of the two directions is proved by means of an inductive procedure. We start by recalling the rules of both calculi (Section 5.2); then, we present the "easy" direction of the translation (Section 5.3) followed by the "difficult" one (Section 5.4).

5.2 Sequent calculi G3V and $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$

Refer to Section 4.1 of the previous chapter for a presentation of logic $\mathbb{V}$. The set of well-formed formulas of $\mathbb{V}$ and its extensions is defined as $\mathcal{F}^{\preccurlyeq} ::= p \mid \bot \mid A \to A \mid A \preccurlyeq A$. Sequent calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ was introduced in section 4.4 of chapter 4. We report the rules in Figure 5.1. Recall that a *block* $[\Sigma \lhd A]$ is a structure interpreted as a disjunction of comparative plausibility formulas: $A \preccurlyeq B_1 \vee \cdots \vee A \preccurlyeq B_n$, for $\Sigma \equiv B_1, \ldots, B_n$. Admissibility of the structural rules of weakening and contraction can be found in Section 4.4. In particular, we will need the rule of weakening inside blocks and on blocks:

$$\frac{\Gamma \Rightarrow \Delta\,[\Sigma \lhd C]}{\Gamma \Rightarrow \Delta\,[A, \Sigma \lhd C]}\ \mathsf{Wk_i} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, [\Sigma \lhd C]}\ \mathsf{Wk_B}$$

As for **G3V**, the sequent calculus can be found in Section 3.7 of Chapter 3. The rules are reported in Figure 5.2, and are defined by spelling out the semantic conditions

$$\textbf{Initial sequents} \quad \frac{}{p,\Gamma \Rightarrow \Delta, p}\ \mathsf{init} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ \bot_{\mathsf{L}}$$

Propositional rules

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ \mathsf{L} \to^{\mathsf{i}} \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ \mathsf{R} \to^{\mathsf{i}}$$

Rules for comparative plausibility

$$\frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B}\ \mathsf{R} \preccurlyeq^{\mathsf{i}}$$

$$\frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd C]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C]}\ \mathsf{L} \preccurlyeq^{\mathsf{i}}$$

Rules for blocks

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}\ \mathsf{com}^{\mathsf{i}}$$

$$\frac{A \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{jump}$$

Figure 5.1: Rules of $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$

associated to the operators in nested neighbourhood models (or, equivalently, sphere models). Recall the truth condition for the comparative plausibility operator:

$$x \Vdash A \preccurlyeq B \quad \textit{iff} \quad \textit{for all } \alpha \in N(x) \textit{ if } \alpha \Vdash^{\exists} B \textit{ then } \alpha \Vdash^{\exists} A.$$

Proof of height-preserving admissibility of weakening and contraction can be found in Section 3.7 of Chapter 3. We need admissibility of the following additional rule.

Lemma 5.2.1. Rule $\mathsf{Mon}\exists$ is admissible.

$$\frac{b \subseteq a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A, b \Vdash^{\exists} A}{b \subseteq a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{Mon}\exists$$

Proof. By induction on the height h of the derivations of the premisses. If $h = 0$, the premiss of the rule is an initial sequent, also its conclusion is an initial sequent. Similarly if it is conclusion of $\bot_{\mathsf{L}}$. If $h > 0$, we distinguish cases according to the last rule applied to derive the premiss. If the last rule applied is a rule in which $b \Vdash^{\exists} A$ is *not* the principal formula, it suffices to apply the inductive hypothesis to the premiss of the rule and then apply the rule again. For instance, suppose that the last rule applied is $\mathsf{R} \Vdash^{\exists}$, applied to $a \Vdash^{\exists} A$.

$$\frac{b \subseteq a, x \in a, \Gamma' \Rightarrow \Delta, a \Vdash^{\exists} A, b \Vdash^{\exists} A, x : A}{b \subseteq a, x \in a, \Gamma' \Rightarrow \Delta, a \Vdash^{\exists} A, b \Vdash^{\exists} A}\ \mathsf{R} \Vdash^{\exists}$$

By inductive hypothesis, the sequent $b \subseteq a, x \in a, \Gamma' \Rightarrow \Delta, a \Vdash^{\exists} A, x : A$ is derivable; if we apply $\mathsf{R} \Vdash^{\exists}$ to it we obtain the desired conclusion $b \subseteq a, x \in a, \Gamma' \Rightarrow \Delta, a \Vdash^{\exists} A$.

If the premiss of $\mathsf{Mon}\exists$ has been derived by a rule in which $b \Vdash^{\exists} A$ is the principal formula, it has been derived by $\mathsf{R} \Vdash^{\exists}$ applied to $b \Vdash^{\exists} A$.

$$\frac{b \subseteq a, x \in b, \Gamma' \Rightarrow \Delta, a \Vdash^{\exists} A, b \Vdash^{\exists} A, x : A}{b \subseteq a, x \in b, \Gamma' \Rightarrow \Delta, a \Vdash^{\exists} A, b \Vdash^{\exists} A}\ \mathsf{R} \Vdash^{\exists}$$

Initial sequents $\overline{x : p, \Gamma \Rightarrow \Delta, x : p}$ init $\quad \overline{x : \bot, \Gamma \Rightarrow \Delta}$ $\bot_{\mathsf{L}}$

Rules for local forcing

$$\frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ \mathsf{L} \Vdash^{\exists} (\mathsf{x!}) \qquad \frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{R} \Vdash^{\exists}$$

Propositional rules

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \to B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\to \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \to B}\ \mathsf{R}\to$$

Rules for comparative plausibility

$$\frac{a \Vdash^{\exists} B, a \in N(x), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : A \preccurlyeq B}\ \mathsf{R} \preccurlyeq (\mathsf{a!})$$

$$\frac{a \in N(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} B \quad a \Vdash^{\exists} A, a \in N(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta}{a \in N(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta}\ \mathsf{L} \preccurlyeq$$

Rules for inclusion and nesting

$$\frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ \mathsf{L} \subseteq$$

$$\frac{a \subseteq b, a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta \qquad b \subseteq a, a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}\ \mathsf{Nes}$$

Figure 5.2: Rules of **G3V**

By inductive hypothesis, the sequent $b \subseteq a, x \in b, \Gamma' \Rightarrow \Delta, a \Vdash^{\exists} A, x : A$ is derivable. We proceed with the following transformation:

$$\cfrac{\cfrac{\cfrac{b \subseteq a, x \in b, \Gamma' \Rightarrow \Delta, a \Vdash^{\exists} A, x : A}{x \in a, b \subseteq a, x \in b, \Gamma' \Rightarrow \Delta, a \Vdash^{\exists} A, x : A}\ \mathsf{Wk}}{x \in a, b \subseteq a, x \in b, \Gamma' \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{R} \Vdash^{\exists}}{b \subseteq a, x \in b, \Gamma' \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{L} \subseteq$$

The endsequent is the conclusion of $\mathsf{Mon}\exists$. □

Example 5.2.1. By means of example, we show a derivation of the axiom (CO) $A \preccurlyeq B \vee B \preccurlyeq A$ both in **G3V** and in $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$. In the derivation with **G3V** we have omitted writing the right-hand premiss of Nes, sequent $b \subseteq a, a \in N(x), b \in N(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B$, derivable in a similar way as the left-hand premiss. Rules $\mathsf{R}\vee$ and $\mathsf{R}\vee^{\mathsf{i}}$ are derivable in both calculi.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overset{\triangledown}{a \subseteq b, a \in N(x), b \in N(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, y : B, b \Vdash^{\exists} B}}{a \subseteq b, a \in N(x), b \in N(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\ \mathsf{R} \Vdash^{\exists}}{a \subseteq b, a \in N(x), b \in N(x), y \in a, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\ \mathsf{L} \subseteq}{a \subseteq b, a \in N(x), b \in N(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\ \mathsf{L} \Vdash^{\exists}}{a \in N(x), b \in N(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\ \mathsf{Nes}}{a \in N(x), a \Vdash^{\exists} B \Rightarrow a \Vdash^{\exists} A, x : B \preccurlyeq A}\ \mathsf{R} \preccurlyeq}{\Rightarrow x : A \preccurlyeq B, x : B \preccurlyeq A}\ \mathsf{R} \preccurlyeq}{\Rightarrow x : A \preccurlyeq B \vee B \preccurlyeq A}\ \mathsf{R}\vee$$

$$
\dfrac{\dfrac{\overset{\triangledown}{B \Rightarrow A, B}}{\Rightarrow [A, B \lhd B], [B \lhd A]}\ \mathsf{jump} \qquad \dfrac{\overset{\triangledown}{A \Rightarrow A, B}}{\Rightarrow [A \lhd B], [A, B \lhd A]}\ \mathsf{jump}}{\dfrac{\dfrac{\dfrac{\Rightarrow [A \lhd B], [B \lhd A]}{\Rightarrow [A \lhd B], B \preccurlyeq A}\ \mathsf{R}\preccurlyeq^{\mathsf{i}}}{\Rightarrow A \preccurlyeq B, B \preccurlyeq A}\ \mathsf{R}\preccurlyeq^{\mathsf{i}}}{\Rightarrow A \preccurlyeq B \vee B \preccurlyeq A}\ \mathsf{R}\vee^{\mathsf{i}}}\ \mathsf{com}^{\mathsf{i}}
$$

In the rest of the chapter we adopt the following notation:

Remark 5.2.1. Notational convention: given multisets of formulas $\Gamma = \{A_1, \ldots, A_m\}$ and $\Sigma = \{S_1, \ldots, S_k\}$, we shall write $x : \Gamma$ and $a \Vdash^{\exists} \Sigma$ as abbreviations for $x : A_1, \ldots, x : A_m$ and $a \Vdash^{\exists} S_1, \ldots, a \Vdash^{\exists} S_k$ respectively.

Since we aim at translating derivations, we introduce a notation to represent derivations, applicable to both calculi.

Definition 5.2.1. Let INIT be a **G3V** / $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ initial sequent, and let SEQ denote a **G3V** / $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ sequent $\Gamma \Rightarrow \Delta$. Let R be a **G3V** / $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ rule. A derivation is the following object, where (1) and (2) are sequents:

$$
\mathcal{D} \ : \quad \overset{\triangledown}{\text{INIT}} \quad ; \quad \dfrac{\begin{matrix}\mathcal{D}_1 \\ (1)\end{matrix}}{\text{SEQ}}\ R \quad ; \quad \dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2 \\ (1) & (2)\end{matrix}}{\text{SEQ}}\ R
$$

5.3 From the internal to the labelled sequent calculus

In this section, we show how $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivations can be translated into derivations in **G3V** + Wk + Ctr + Mon$\exists$. We first define t, a translation for sequents; then, we specify a function which takes as argument derivations in the internal sequent calculus and produces in output derivations in the labelled one. Finally, we prove that the translation specified by the function is correct.

Definition 5.3.1. Given a world label x, a list of countably many neighbourhood labels $\bar{a} = a_1\, a_2\, \ldots a_n$ and multisets of formulas $\Gamma, \Delta, \Sigma_1, \ldots, \Sigma_n$, define:

- $t(\Gamma)^x \ := \ x : F_1, ..., x : F_k$
- $t(\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd B_1], ..., [\Sigma_n \lhd B_n])^{x,\bar{a}} :=$

  $a_1, ..., a_n \in N(x),\ a_1 \Vdash^{\exists} B_1, ..., a_n \Vdash^{\exists} B_n, t(\Gamma)^x \Rightarrow t(\Delta)^x, a_1 \Vdash^{\exists} \Sigma_1, ..., a_n \Vdash^{\exists} \Sigma_n$

The translation takes as parameter one world label, x, and neighbourhood labels $\bar{a}$: the idea is that for each block $[\Sigma_i \lhd B_i]$ we introduce a new neighbourhood label a_i such that $a_i \in N(x)$, and formulas $a_i \Vdash^{\exists} B_i$ in the antecedent and $a_i \Vdash^{\exists} \Sigma_i$ in the consequent. These formulas correspond to the semantic condition for a block i.e., a disjunction of comparative plausibility formulas in sphere models:

$$x \Vdash [\Sigma \lhd B] \quad \textit{iff} \quad \textit{for all } \alpha \in N(x) \textit{ if } \alpha \Vdash^{\exists} B \textit{ then } \alpha \Vdash^{\exists} \Sigma.$$

Figure 5.3 defines a function $\{\ \}^{x,\bar{a}}$ that takes as input a $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivation $\mathcal{D}$ and produces in output a **G3V** derivation $\{\mathcal{D}\}^{x,\bar{a}}$. The parameters of the function are the labels x and $\bar{a}$; these are the world and neighbourhood labels used to translate the root sequent of $\mathcal{D}$. For $\bar{a} = (a_1 \ldots a_n)$, we write $\bar{a}b$ to denote the list $(a_1 \ldots a_n\ b)$. For a rule R of a calculus, we denote by $R(n)$ n applications of R. The function for propositional rules is immediate: from a translation of the premiss(es) derive a translation of the conclusion applying the corresponding **G3V** rule.

(init) $$\left\{\overset{\triangledown}{\mathrm{INIT}}\right\}^{x,\bar{a}} \rightsquigarrow \; t(\overset{\triangledown}{\mathrm{INIT}})^{x,\bar{a}}$$

(R $\preccurlyeq^{\mathsf{i}}$) $$\left\{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow \Delta, [A \lhd B]\end{array}}{\Gamma \Rightarrow \Delta, A \preccurlyeq B}\,\mathsf{R}\preccurlyeq^{\mathsf{i}}\right\}^{x,\bar{a}} \rightsquigarrow \quad \dfrac{\begin{array}{c}\{\mathcal{D}_1\}^{x,\bar{a}\,b}\\ t(\Gamma \Rightarrow \Delta, [A \lhd B])^{x,\bar{a}\,b}\end{array}}{t(\Gamma \Rightarrow \Delta, A \preccurlyeq B)^{x,\bar{a}}}\,\mathsf{R}\preccurlyeq$$

(L $\preccurlyeq^{\mathsf{i}}$) $$\left\{\dfrac{\begin{array}{c}\mathcal{D}_1\\ A \preccurlyeq B, \Gamma \Rightarrow \Delta\,[\Sigma, B \lhd C]\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\Sigma \lhd A]\,, [\Sigma \lhd C]\end{array}}{A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\,\mathsf{L}\preccurlyeq^{\mathsf{i}}\right\}^{x,\bar{a}\,b} \rightsquigarrow$$

$$\dfrac{\begin{array}{c}\{\mathcal{D}_1\}^{x,\bar{a}\,b}\\ t(A \preccurlyeq B, \Gamma \Rightarrow \Delta\,[\Sigma, B \lhd C])^{x,\bar{a}\,b}\end{array} \quad \dfrac{\begin{array}{c}\{\mathcal{D}_2\}^{x,\bar{a}\,b\,c\,[c/b]}\\ t(A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\Sigma \lhd A]\,, [\Sigma \lhd C])^{x,\bar{a}\,b\,c\,[c/b]}\end{array}}{b \in N(x), b \Vdash^{\exists} A, b \Vdash^{\exists} C, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma}\,\mathsf{Ctr}}{t(A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\Sigma \lhd A])^{x,\bar{a}\,b}}\,\mathsf{L}\preccurlyeq$$

(com$^{\mathsf{i}}$) $$\left\{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A]\,, [\Sigma_2 \lhd B]\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma \Rightarrow \Delta, [\Sigma \lhd A]\,, [\Sigma, \Pi \lhd B]\end{array}}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A]\,, [\Sigma_1 \lhd B]}\,\mathsf{com}^{\mathsf{i}}\right\}^{x,\bar{a}\,b\,c} \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\{\mathcal{D}_1\}^{x,\bar{a}\,b\,c}\\ t(\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A]\,, [\Sigma_2 \lhd B])^{x,\bar{a}\,b\,c}\end{array}}{b \subseteq c, b \Vdash^{\exists} A, c \Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma_1, b \Vdash^{\exists} \Sigma_2, c \Vdash^{\exists} \Sigma_2}\,\mathsf{Wk}}{b \subseteq c, b \Vdash^{\exists} A, c \Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma_1, c \Vdash^{\exists} \Sigma_2}\,\mathsf{Mon}\exists \quad \dfrac{\dfrac{\begin{array}{c}\{\mathcal{D}_2\}^{x,\bar{a}\,b\,c}\\ t(2)^{x,\bar{a}\,b\,c}\end{array}}{(2')}\,\mathsf{Wk}}{(2'')}\,\mathsf{Mon}\exists}{t(\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A]\,, [\Sigma_1 \lhd B])^{x,\bar{a}\,b\,c}}\,\mathsf{Nes}$$

(2) $\Gamma \Rightarrow \Delta, [\Sigma \lhd A]\,, [\Sigma, \Pi \lhd B]$;
(2′) $b \subseteq c, b \Vdash^{\exists} A, c \Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma_1, c \Vdash^{\exists} \Sigma_1, c \Vdash^{\exists} \Sigma_2$;
(2″) $c \subseteq b, b \Vdash^{\exists} A, c \Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma_1, c \Vdash^{\exists} \Sigma_2$

(jump)

$$\left\{\dfrac{\begin{array}{c}\mathcal{D}_1\\ x : \Sigma \Rightarrow x : A\end{array}}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\,\mathsf{jump}\right\}^{x,\bar{a}\,b} \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}t\{\mathcal{D}_1\}^{x\,[x/y]}\\ t(x : \Sigma \Rightarrow x : A)^{x\,[x/y]}\end{array}}{y \in b, b \in N(x), y : A, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, y : \Sigma, b \Vdash^{\exists} \Sigma}\,Wk}{y \in b, b \in N(x), y : A, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma}\,\mathsf{R}\Vdash^{\exists}(n)}{t(\Gamma \Rightarrow \Delta, [\Sigma \lhd A])^{x,\bar{a}\,b}}\,\mathsf{L}\Vdash^{\exists}$$

Figure 5.3: From $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ to **G3V** derivations

Theorem 5.3.1. Let $\mathcal{D}$ be a $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivation of $\Gamma \Rightarrow \Delta$. Then $\{\mathcal{D}\}^{x,\bar{a}}$ is a derivation of $t(\Gamma \Rightarrow \Delta)^{x,\bar{a}}$ in **G3V**.

Proof. By induction on the height h of the derivation of the sequent. If $h = 0$, $\Gamma \Rightarrow \Delta$ is a $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ initial sequent, and $t(\Gamma \Rightarrow \Delta)^{x,\bar{a}}$ is a **G3V** initial sequent. If $h > 0$, $\Gamma \Rightarrow \Delta$ must have been derived applying a rule of $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$. All cases easily follow applying the clauses of the procedure described above.

[$\mathsf{R} \preccurlyeq^{\mathsf{i}}$] The translation of the premiss of $\mathsf{R} \preccurlyeq^{\mathsf{i}}$ is the **G3V** sequent $t(\Gamma \Rightarrow \Delta, A \preccurlyeq B)^{x,\bar{a}} = b \in N(x), b \Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} A$. Applying $\mathsf{R} \preccurlyeq$ we obtain the translation of the conclusion: $t(\Gamma \Rightarrow \Delta, A \preccurlyeq B)^{x,\bar{a}} = t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, x : A \preccurlyeq B$.

[$\mathsf{L} \preccurlyeq^{\mathsf{i}}$] The translations of the premisses are the **G3V** sequents: $t(A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\Sigma, B \lhd C])^{x,\bar{a}\,b} = b \in N(x), b \Vdash^{\exists} C, x : A \preccurlyeq B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma, b \Vdash^{\exists} B$; $t(A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd C])^{x,\bar{a}\,b\,c} = b \in N(x), c \in N(x), b \Vdash^{\exists} A, c \Vdash^{\exists} C, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma, c \Vdash^{\exists} \Sigma$. We substitute the neighbourhood label c with b in the second sequent, obtaining $b \in N(x), b \in N(x), b \Vdash^{\exists} A, c \Vdash^{\exists} C, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma, b \Vdash^{\exists} \Sigma$. After application of contraction, application of $\mathsf{L} \preccurlyeq$ to this sequent and to the translation of the first premiss yields sequent $b \in N(x), b \Vdash^{\exists} C, x : A \preccurlyeq B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma$. This sequent is the translation of the $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ sequent $A \preccurlyeq B, \Gamma \Rightarrow \Delta, [\Sigma \lhd C]$, with parameters $x, \bar{a}\,b$.

[$\mathsf{com}^{\mathsf{i}}$] The translations of the premisses are the **G3V** sequents:

$t(\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B])^{x,\bar{a}\,b\,c} = b \in N(x), c \in N(x), b \Vdash^{\exists} A, c \Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma_1, b \Vdash^{\exists} \Sigma_2, c \Vdash^{\exists} \Sigma_2$;

$t(\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B])^{x,\bar{a}\,b\,c} =$
$b \in N(x), c \in N(x), b \Vdash^{\exists} A, c \Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma_1, c \Vdash^{\exists} \Sigma_2, c \Vdash^{\exists} \Sigma_2$.

We add by weakening $b \subseteq c$ to the translation of the first premiss and $c \subseteq b$ to the translation of the second, and apply rule $\mathsf{Mon}\exists$ to both. A final application of Nes yields the desired sequent: $t(\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd b])^{x,\bar{a}\,b\,c} = b \in N(x), c \in N(x), b \Vdash^{\exists} A, c \Vdash^{\exists} B, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma_1, c \Vdash^{\exists} \Sigma_2$.

[jump] The translation of the premiss of jump is the sequent $t(x : \Sigma \Rightarrow x : A)^{x} = x : \Sigma \Rightarrow x : A$. We substitute x with a fresh world label y, and apply the transformations described above; we obtain the sequent $b \in N(x), b \Vdash^{\exists} A, t(\Gamma)^{x,\bar{a}} \Rightarrow t(\Delta)^{x,\bar{a}}, b \Vdash^{\exists} \Sigma$, which is the translation of the sequent $\Gamma \Rightarrow \Delta, [\Sigma \lhd A]$, the conclusion of jump, with parameters $x, \bar{a}\,b$. $\square$

Example 5.3.1. By means of example, we show how the $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivation of axiom (CO) (Example 5.2.1) is translated into a **G3V** derivation. We show only the left premiss of the application of Nes.

$$
\begin{prooftree}
\AxiomC{$\triangledown$}
\noLine
\UnaryInfC{$y : B \Rightarrow y : A, y : B$}
\RightLabel{Wk}
\UnaryInfC{$a \in N(x), b \in N(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B, y : A, y : B$}
\RightLabel{$\mathsf{R} \Vdash^{\exists}(2)$}
\UnaryInfC{$a \in N(x), b \in N(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B$}
\RightLabel{$\mathsf{L} \Vdash^{\exists}$}
\UnaryInfC{$a \in N(x), b \in N(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B$}
\RightLabel{Wk}
\UnaryInfC{$a \subseteq b, a \in N(x), b \in N(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B$}
\RightLabel{$\mathsf{Mon}\forall$}
\UnaryInfC{$a \subseteq b, a \in N(x), b \in N(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B$}
\RightLabel{Nes}
\UnaryInfC{$a \in N(x), b \in N(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B$}
\RightLabel{$\mathsf{R} \preccurlyeq^{\mathsf{i}}$}
\UnaryInfC{$a \in N(x), a \Vdash^{\exists} B \Rightarrow x : B \preccurlyeq A, a \Vdash^{\exists} A$}
\RightLabel{$\mathsf{R} \preccurlyeq^{\mathsf{i}}$}
\UnaryInfC{$\Rightarrow x : A \preccurlyeq B, x : B \preccurlyeq A$}
\RightLabel{$\mathsf{R}\vee$}
\UnaryInfC{$\Rightarrow x : A \preccurlyeq B \vee B \preccurlyeq A$}
\end{prooftree}
$$

5.4 From the labelled to the internal calculus

The inverse translation takes care of translating **G3V** derivations into $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivations. With respect to the previous translation, we are faced with an additional difficulty: there are **G3V** derivable sequents that cannot be translated into $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ sequents or, equivalently, there are *more* **G3V** derivable sequents than $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivable sequents. For this reason, proving that if $t(\Gamma \Rightarrow \Delta)^x$ is derivable in **G3V** then $\Gamma \Rightarrow \Delta$ is derivable in $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ would not work: in the **G3V** derivation of $t(\Gamma \Rightarrow \Delta)^x$ there could occur some sequents that are not in the range of the translation t.

Thus, we need a more complex proof strategy. After defining a translation s for sequents, we introduce the notion of normal form derivations in **G3V**: the idea is that we cannot translate *any* derivation, but only those constructed following a certain order of application of the rules. We first prove that any derivation in **G3V** can be transformed into a normal form derivation. Then, we prove the Jump lemma, which allows us to "skip" the sequents that cannot be translated in the derivation. Finally, we define a function $[\![\,]\!]$ to translate **G3V** derivations into $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivations.

Definition 5.4.1. Let $\Gamma \Rightarrow \Delta$ be a sequent of the form

$$\begin{gathered}\mathcal{R}^{a_1}, \dots, \mathcal{R}^{a_n}, a_1 \in N(x), ..., a_n \in N(x), a_1 \Vdash^{\exists} A_1, ..., a_n \Vdash^{\exists} A_n, x : \Gamma^P \\ \Rightarrow x : \Delta^P, a_1 \Vdash^{\exists} \Sigma_1, ..., a_n \Vdash^{\exists} \Sigma_n\end{gathered}$$

where:

a) each $\mathcal{R}^{a_i}$ contains zero or more inclusions $a_i \subseteq a_j$ for $1 \leqslant i \leqslant j \leqslant n$;
b) Γ^P and Δ^P are composed only of propositional and $\preccurlyeq$ formulas;
c) for each a_i, there is exactly one formula $a_i \Vdash^{\exists} A_i$ in the antecedent, and at least one formula $a_i \Vdash^{\exists} B_i$ in the consequent.

The translation s takes as parameter world label x, label of Γ^P and Δ^P, and is defined as

$$s(\Gamma \Rightarrow \Delta)^x := \Gamma^P \Rightarrow \Delta^P, \Pi$$

where: Γ^P is obtained from $x : \Gamma^P$ by removing the label x, Δ^P is obtained from $x : \Delta^P$ by removing the label x, and Π contains n blocks

$$[\{\Sigma_1\} \lhd A_1], ..., [\{\Sigma_n\} \lhd A_n]$$

where each $\{\Sigma_i\}$ is the multiset union $\{\Sigma_i\} = \Sigma_i \cup \bigcup \{\Sigma_j \mid a_i \subseteq a_j \textit{ occurs in } \mathcal{R}^{a_i}\}$.

Example 5.4.1. The sequent

$$P = a_1 \subseteq a_2, a_2 \subseteq a_3, a_1 \subseteq a_3, a_1 \in N(x), a_2 \in N(x), a_3 \in N(x), a_1 \Vdash^{\exists} A_1, a_2 \Vdash^{\exists} A_2,$$
$$a_3 \Vdash^{\exists} A_3, x : \Gamma \Rightarrow x : \Delta, a_1 \Vdash^{\exists} \Sigma_1, a_2 \Vdash^{\exists} \Sigma_2, a_3 \Vdash^{\exists} \Sigma_3$$

is translated as follows:

$$s(P)^x \;=\; s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma_1, \Sigma_2, \Sigma_3 \lhd A_1], [\Sigma_2, \Sigma_3 \lhd A_2], [\Sigma_3 \lhd A_3].$$

Intuitively, s re-assembles the blocks from formulas labelled with the same neighbourhood label. Furthermore, for each inclusion $a_i \subseteq a_j$ we add to the corresponding block also formulas Σ_j such that $a_j \Vdash^{\exists} \Sigma_j$ occurs in the consequent of the labelled sequent[1]. Thus, each block in the internal calculus consists of $\preccurlyeq$-formulas relative to some sphere i.e., labelled with the same neighbourhood label in **G3V**.
We now introduce the notion of normal form derivations and state the Jump lemma.

[1]The intuition is that comparative plausibility formulas in the consequent (i.e., blocks) pass from larger to smaller spheres. If we interpret formulas in the consequent as false (tableaux interpretation) the requirement is saying that if a comparative plausibility formula is false at a sphere α_j, then it is false also at smaller spheres α_i.

Definition 5.4.2. Given a world label x and a sequent $\Gamma \Rightarrow \Delta$, the sequent is *saturated with respect to variable* x (is x-saturated) if:

a) if $x : A$ belongs to $\Gamma \cup \Delta$, then A is atomic or $A \equiv B \preccurlyeq C$ and $x : B \preccurlyeq C$ does not belong to Δ;
b) if $x : A \preccurlyeq B$ and $a \in N(x)$ occur in Γ, either $a \Vdash^{\exists} A$ occurs in Δ or $a \Vdash^{\exists} B$ occurs in Γ;
c) if $a \in N(x)$ and $b \in N(x)$ occur in Γ, either $a \subseteq b$ or $b \subseteq a$ occurs in Γ.

A sequent is *x-hypersaturated with respect to variable x* if, for all $a \in N(x)$, the following hold:

a) for each $a \in N(x)$, no formulas $a \Vdash^{\exists} B$ occurs in Γ;
b) for each $a \in N(x)$, $y \in a$ occurring in Γ and for each $a \Vdash^{\exists} B$ occurring in Δ, there is a formula $y : B$ occurring in Δ;
c) for each $a \in N(x)$, $b \in N(x)$, $a \subseteq b$ and $y \in a$ occurring in Γ, there is a formula $y \in b$ occurring in Γ.

Given a branch B of a derivation of $\Gamma \Rightarrow \Delta$, we say that B is in normal form with respect to x if from the root sequent $\Gamma \Rightarrow \Delta$ upwards the following holds: first all propositional and $\preccurlyeq$ rules are applied until an x-saturated sequent is reached; then rules $\mathsf{R} \Vdash^{\exists}$, $\mathsf{L} \Vdash^{\exists}$ and $\mathsf{L} \subseteq$ are applied to the x-saturated sequent until a x-hypersaturated is reached. We say that a derivation of $\Gamma \Rightarrow \Delta$ is *in normal form* with respect to x if all its branches are in normal form.

The following definition identifies the structure of labels in the derivation, similarly as done in Definition 3.5.4 of Chapter 3.

Definition 5.4.3. Given a branch $\mathrm{B} = S_0, S_1, \ldots$ where $S_i = \Gamma_i \Rightarrow \Delta_i$ for $i = 1, 2, \ldots$ and let $\Pi_{\mathrm{B}} = \bigcup_i \Gamma_i$. We define the following relations:

- $x \to_g a$ if $a \in N(x)$ occurs in Π_{B};
- $a \to_g y$ if for some $S_i = \Gamma_i \Rightarrow \Delta_i$, $y \in a$ occurs in Γ_i and y does not occur in any S_j with $j < i$;
- $x \xrightarrow{w} y$ if there exists an a such that $x \to_{\Pi_{\mathrm{B}}} a$ and $a \to_{\mathrm{B}} y$;
- $x \xrightarrow{w}^* y$ is the transitive closure of $x \to_{\Pi_{\mathrm{B}}} y$.

Remark 5.4.1. By inspection on the rules of **G3V**, observe that if we start the derivation with a sequent $\Rightarrow x : A$, the relation $\xrightarrow{w}$ forms a tree with respect to the formulas occurring in any branch. We can generalize this fact as follows. Let $\Gamma \Rightarrow \Delta$ be a **G3V** derivable sequent obtained as the result of a translation t - i.e., such that there exists a $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ sequent $\Gamma^I \Rightarrow \Delta^I$ such that $t(\Gamma^I \Rightarrow \Delta^I)^{x,\bar{a}} = \Gamma \Rightarrow \Delta$. Then this sequent will display at most one world label x and possibly different neighbourhood labels $a \in N(x)$. Again, by inspection of the rules we have that, in any branch of the **G3V** derivation for such a sequent, the labels form a tree with respect to the relation $\xrightarrow{w}$. This result is not unexpected: as in the case of [40], we are able to translate only tree-form sequents.

Lemma 5.4.1. Given a **G3V** derivable sequent $\Gamma \Rightarrow \Delta$ which is the result of a translation t and a variable x occurring in it, we can transform any derivation of $\Gamma \Rightarrow \Delta$ into a derivation in normal form with respect to variable x.

Proof. Induction on the height of the derivation of $\Gamma \Rightarrow \Delta$. Let $\xrightarrow{w}^*$ be the relation between world labels occurring in the union of all antecedents of a branch, as in Definition 5.4.3. If the sequent is an axiom, we are done. If the height of the derivation is greater than zero, we proceed by cases: if there are no labels y different from x such that $x \to^*_{\Pi_{\mathrm{B}}} y$, then x is the only label in the branch. The derivation of $\Gamma \Rightarrow \Delta$ will use only propositional rules; thus, the branch is in normal form with respect to x. If there is some label y such that $x \to^*_{\Pi_B} y$, transform each branch of the derivation of $\Gamma \Rightarrow \Delta$ as follows. Sequent $\Gamma \Rightarrow \Delta$ contains at most one world label and possibly some neighbourhood labels $a, b \ldots$ such that $a \in N(x), b \in N(x) \ldots$. Labels $y \in a$ might be introduced only by $\mathsf{L} \Vdash^\exists$. If some rules are applied to formulas $a \Vdash^\exists A$ or to formulas $y \in a, a \subseteq b$ or to formulas $y : A$, when there are still some rules (non-redundantly) applicable to formulas $x : A$ or $a \in N(x), b \in N(x)$, apply *first* the rules for $x : A$ and $a \in N(x), b \in N(x)$, until the x-saturated sequent is reached. Similarly, if some rules are applied to a formula $y : A$ when there are still some rules which can be (non-redundantly) applied to formulas $a \Vdash^\exists A$ or to formulas $y \in a, a \subseteq b$, apply these latter rules until an x-hypersaturated sequent is reached, before proceeding to apply rules for $y : A$. In both cases, permuting the rules in the derivation does not represent a problem: rules applicable to $x : A$ and to $y : A$ involve different active formulas. As for rules applicable to $a \Vdash^\exists A$ with $a \in N(x)$, observe that the normal form "respects" the order in which labels are generated in the tree. For instance, the normal form with respect to x requires that rule $\mathsf{R} \preccurlyeq$, generating neighbourhood label $a \in N(x)$, has to be applied to $x : C \preccurlyeq D$ *before* rules $\mathsf{R} \Vdash^\exists$ or $\mathsf{L} \Vdash^\exists$ might be applied to $b \Vdash^\exists C$ or $b \Vdash^\exists D$. To obtain a normal form derivation with respect to x, we have to apply the procedure to all branches; we might also have to add some rules to obtain the x-saturated and x-hypersaturated sequents. □

Definition 5.4.4. We now give a simplified version of Definition 5.4.3, needed to prove the Jump lemma. Given a multiset of labelled formulas Π, define:

a) $x \to_\Pi a$ if $a \in N(x)$ occurs in Π;
b) $a \to_\Pi y$ if $y \in a$ occurs in Π;
c) $x \to_\Pi y$ if there exists an a such that $x \to_\Pi a$ and $a \to_\Pi y$ occur in Π.

Let $W_\Pi(x)$ be the reflexive and transitive closure of $x \to_\Pi y$: $W_\Pi(x) = \{y \mid x \to^*_\Pi y\}$. Let $N_\Pi(x) = \{b \mid \exists u \,.\, x \to^*_\Pi u \textit{ and } u \to_\Pi b\}$. These sets represents respectively the set of world labels accessible from a world label x, and the set of neighbourhood labels accessible from a world label x occurring in Π. Define Σ^Π_x as the union of the sets:

$$\begin{aligned} \Sigma^\Pi_x \;=\; & \{u : F \mid u : F \textit{ occurs in } \Sigma \textit{ and } u \in W_\Pi(x)\} \cup \\ & \cup \{a \Vdash^\exists B \mid a \Vdash^\exists B \textit{ occurs in } \Sigma \textit{ and } a \in N_\Pi(x)\} \cup \\ & \cup \{b \in S(y) \mid b \in S(y) \textit{ occurrs in } \Pi,\ b \in N_\Pi(x) \textit{ and } y \in W_\Pi(x)\} \cup \\ & \cup \{a \subseteq b \mid a \subseteq b \textit{ occurs in } \Pi \textit{ and } a, b \in N_\Pi(x)\} \cup \\ & \cup \{z \in a \mid z \in a \textit{ occurs in } \Pi \textit{ and } a \in N_\Pi(x)\}. \end{aligned}$$

Lemma 5.4.2 (Jump lemma)**.** Let $\Gamma \Rightarrow \Delta$ be a derivable **G3V** sequent. If the labels occurring in $W_\Gamma(x)$ have a tree structure, for each label x occurring in the sequent, it holds that either 1) sequent $\Gamma^\Gamma_x \Rightarrow \Delta^\Gamma_x$ or 2) sequent $\Gamma - \Gamma^\Gamma_x \Rightarrow \Delta - \Delta^\Gamma_x$ is derivable, with the same derivation height.

Proof. To simplify the notation, we write Γ^* for Γ^Γ_x and Δ^* for Δ^Γ_x respectively. The proof is by induction on the height of the derivation, and by distinction of cases. If $\Gamma \Rightarrow \Delta$ is an initial sequent, it has the form $u : P, \Gamma' \Rightarrow \Delta', u : P$. If $u \in W^\Gamma_x$, we have that $u : P, \Gamma^* \Rightarrow \Delta^*, u : P$ is and initial sequent, hence derivable, and we are in case 1. If $u \notin W^\Gamma_x$, then $u : P \in \Gamma - \Gamma^\Gamma_x$, and we obtain case 2. For the propositional rules, we show only the case of $L \to$.

$$\frac{\Gamma \Rightarrow \Delta, u : B \quad u : A, \Gamma \Rightarrow \Delta}{u : A \to B, \Gamma \Rightarrow \Delta}\; L \to$$

Suppose $u \in W^\Gamma_x$. We have to show that either 1) the sequent $u : A \to B, \Gamma^* \Rightarrow \Delta^*$ is derivable, or that 2) sequent $\Gamma - \Gamma^* \Rightarrow \Delta - \Delta^*$ is derivable. By inductive hypothesis applied to both premisses, we have that either a) $\Gamma^* \Rightarrow \Delta^*, u : B$ is derivable or b)

$\Gamma - \Gamma^* \Rightarrow \Delta - \Delta^*$ is derivable, and that either c) $u : A, \Gamma^* \Rightarrow \Delta^*$ is derivable or d) $\Gamma - \Gamma^* \Rightarrow \Delta - \Delta^*$ is derivable. If *a*) and *c*) are derivable, we apply $L \rightarrow$ and obtain the derivable sequent $u : A \rightarrow B, \Gamma^* \Rightarrow \Delta^*$ (which is case 1). If *a*) and *d*) are derivable, *d*) is already the sequent corresponding to case 2 of the statement; the same holds if *b*) and *c*) are derivable, and if *b*) and *d*) are derivable. If $u \notin W^\Gamma_x$, we want to show that either 1) $\Gamma^* \Rightarrow \Delta^*$ id derivable or that 2) $\Gamma - \Gamma^*, u : A \rightarrow B \Rightarrow \Delta - \Delta^*$ is derivable. Again, by inductive hypothesis we have that either *a*) $\Gamma - \Gamma^*, u : B \Rightarrow \Delta - \Delta^*$ or *b*) $\Gamma^* \Rightarrow \Delta^*$ and either *c*) $\Gamma - \Gamma^* \Rightarrow \Delta - \Delta^*, u : A$ or *d*) $\Gamma^* \Rightarrow \Delta^*$ are derivable. If *a*) and *c*) are derivable, we obtain case 2; otherwise we are in case 1.
If $\Gamma \Rightarrow \Delta$ has been derived by $\mathsf{L} \Vdash^\exists$, we have:

$$\frac{\Gamma, y \in a, y : B \Rightarrow \Delta}{\Gamma, a \Vdash^\exists B \Rightarrow \Delta} \; \mathsf{L} \Vdash^\exists$$

where y does not occur in Γ and Δ. If $a \in N_\Gamma(x)$ then $y \in W_\Gamma(x)$; by inductive hypothesis, we have that either *a*) $\Gamma^*, y : B, y \in a \Rightarrow \Delta^*$ is derivable, or *b*) $\Gamma - \Gamma^* \Rightarrow \Delta - \Delta *$ is derivable. In the former case, a step of $\mathsf{L} \Vdash^\exists$ gives that $\Gamma^*, a \Vdash^\exists B \Rightarrow \Delta^*$ is derivable. If *b*) is derivable, we already have our desired sequent (case 2). If $a \notin N_\Gamma(x)$, then $y \notin W_\Gamma(x)$. By inductive hypothesis, either $\Gamma^* \Rightarrow \Delta^*$ is derivable, and we are done, or $\Gamma - \Gamma^*, y : B, y \in a \Rightarrow \Delta - \Delta^*$ is derivable. Apply $\mathsf{L} \Vdash^\exists$ to obtain the sequent $\Gamma - \Gamma^*, a \Vdash^\exists B \Rightarrow \Delta - \Delta^*$.
For the remaining cases: $\mathsf{R} \preccurlyeq$ is similar to $\mathsf{L} \Vdash^\exists$. Rules Nes and $\mathsf{L} \preccurlyeq$ are similar to $L \rightarrow$. $\mathsf{R} \Vdash^\exists$ and $\mathsf{L} \subseteq$ are immediate, since they do not introduce new labels and have just one premiss. □

We have seen that the labels of a **G3V** derivable sequent form the structure of a tree, in the sense explained in Definition 5.4.3. The Jump lemma is saying that, given such a sequent, and given a world label x occurring in it, it holds that either the sub-tree containing x is relevant to derive an axiom, or the rest of the sequent is relevant[2].

Example 5.4.2. Suppose that the following sequent is derivable.

$$a \in N(x), b \in N(x), y \in a, y : B, z \in a, z : C \Rightarrow a \Vdash^\exists A, b \Vdash^\exists B, z : B, y : A$$

Consider label z: $N^\Gamma_z = \varnothing$ and $W^\Gamma_z = \{z\}$. Thus, Γ^Γ_z coincides with $z : C$, and Δ^Γ_z coincides with $z : B$, and either $z : C \Rightarrow z : A$ is derivable, or the rest of the sequent is derivable, namely $a \in N(x), b \in N(x), y \in a, y : B, z \in a \Rightarrow a \Vdash^\exists A, y : A, b \Vdash^\exists B$.

Lemma 5.4.3. If a sequent $a \Vdash^\exists B, a \Vdash^\exists C, \Gamma \Rightarrow \Delta$ is derivable in **G3V**, then either $a \Vdash^\exists B, \Gamma \Rightarrow \Delta$ or $a \Vdash^\exists C, \Gamma \Rightarrow \Delta$ are derivable in **G3V** with same derivation height.

Proof. By induction on the height of the derivation. If $h = 0$ and $a \Vdash^\exists B, a \Vdash^\exists C, \Gamma \Rightarrow \Delta$ is an initial sequent, it must be so in virtue of its propositional part; thus, both $a \Vdash^\exists B, \Gamma \Rightarrow \Delta$ and $a \Vdash^\exists C, \Gamma \Rightarrow \Delta$ are initial sequents. If $h > 0$, we consider the last rule R applied. All the cases are straightforward, by application of the inductive hypothesis to the premiss of R and re-application of R. The only relevant case is $R = \mathsf{L} \Vdash^\exists$, applied to one of the formulas $a \Vdash^\exists A$ or $a \Vdash^\exists B$.

$$\frac{y \in a, y : A, a \Vdash^\exists B, \Gamma \Rightarrow \Delta}{a \Vdash^\exists A, a \Vdash^\exists B, \Gamma \Rightarrow \Delta} \; \mathsf{L} \Vdash^\exists$$

[2] To underline this feature, we could have called Lemma 5.4.2 "Split Lemma". The name "Jump Lemma" refers to the fact that the lemma is used to treat a part of the labelled derivation (from a x-saturated sequent to a x-hypersaturated sequent) which will be replaced by an occurrence of jump in the internal system.

Apply the Jump lemma to the premiss, obtaining that either $y : A \Rightarrow$ is derivable, or $y \in a, a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta$ is derivable. In the former case, apply weakening and $\mathsf{L} \Vdash^{\exists}$ to obtain a derivation of $a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta$. In the latter case, by invertibility of $\mathsf{L} \Vdash^{\exists}$ we have that sequent $y \in a, w \in a, w : B, \Gamma \Rightarrow \Delta$ is derivable, for some $w \notin \Gamma, \Delta$. We substitute variable y with variable w. The substitution does not affect other formulas than $y \in a$, since $y, w \notin \Gamma, \Delta$. Contraction and $\mathsf{L} \Vdash^{\exists}$ give the sequent $a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta$. $\square$

In order to define a translation $[\![\ ]\!]^x$ for **G3V** normal form derivations we need to define a sub-translation $[\ ]^x$, that takes care of the translation of a derivation from the root sequent up to the x-saturated sequents. This part of the translation is relatively easy, since all labelled sequents occurring in this part of the derivation can be translated into internal sequents. Theorem 5.4.5 will take care of translating the upper part of normal form derivations: from x-saturated sequents to x-hypersaturated sequents, making an essential use of the Jump lemma.

The translation $[\ ]^x$ is defined in Figure 5.4. It takes as parameter a world label x, the one used to translate the root sequent. For $\mathsf{L} \preccurlyeq$ and Nes we explicitly define sets of inclusions that might occur in the sequent: $\mathcal{R}^a = \{a \subseteq c_1, \ldots, a \subseteq c_n\}$; $\mathcal{R}^b = \{b \subseteq d_1, \ldots, b \subseteq d_k\}$. We will also use the corresponding mulisets of formulas:

$$\{\Omega\} = \{c \Vdash^{\exists} \Omega \,|\, c \Vdash^{\exists} \Omega \ \textit{occurs in}\ \Delta\ \textit{and}\ a \subseteq c\ \textit{occurs in}\ \mathcal{R}^a\}$$

$$\{\Xi\} = \{d \Vdash^{\exists} \Xi \,|\, d \Vdash^{\exists} \Xi\ \textit{occurs in}\ \Delta\ \textit{and}\ b \subseteq d\ \textit{occurs in}\ \mathcal{R}^b\}.$$

These sets might be empty, and contain the formulas to be added a block in correspondence to formulas $a \subseteq c$ and $c \Vdash^{\exists} \Omega$ (or $b \subseteq d$ and $d \Vdash^{\exists} \Xi$) occurring in the **G3V** sequent (see Definition 5.4.1).

Lemma 5.4.4. Let $\mathcal{D}^S$ be a **G3V** derivation of $\Gamma \Rightarrow \Delta$ from x-saturated sequents $\Gamma_1^S \Rightarrow \Delta_1^S, \ldots, \Gamma_n^S \Rightarrow \Delta_n^S$; then $[\mathcal{D}^S]^x$ is a derivation of $s(\Gamma \Rightarrow \Delta)^x$ in $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$ from sequents $s(\Gamma_1^{S^-} \Rightarrow \Delta_1^{S^-})^x, \ldots, s(\Gamma_n^{S^-} \Rightarrow \Delta_n^{S^-})^x$, where for each $\Gamma_i^{S^-} \Rightarrow \Delta_i^{S^-}$ it holds that $\Gamma_i^{S^-} \cup \Delta_i^{S^-} \subseteq \Gamma_i^S \cup \Delta_i^S$.

Proof. By distinction of cases, and by induction on the height of the derivation. If $h = 0$, $\Gamma \Rightarrow \Delta$ is a **G3V** initial sequent, and its translation $s(\Gamma \Rightarrow \Delta)$ is a $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$ initial sequent. The propositional cases are obtained applying the corresponding $\mathcal{I}_{\mathbb{V}}^{\mathsf{i}}$ rule to the translation(s) of the premiss(es).

[$\mathsf{R} \preccurlyeq$] Translation of the premiss: $s(a \in N(x), a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A)^x = s(\Gamma)^x \Rightarrow s(\Delta)^x, [A \lhd B]$; translation of the conclusion: $s(\Gamma \Rightarrow \Delta, x : A \preccurlyeq B)^x = s(\Gamma)^x \Rightarrow s(\Delta)^x, A \preccurlyeq B$.

[$\mathsf{L} \preccurlyeq$] $s(1)^x = A \preccurlyeq B, s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, B, \{\Omega\} \lhd C]$. The right premiss (2) cannot be translated, since it features in the antecedent *two* formulas with the same neighbourhood label: $a \Vdash^{\exists} A$ and $a \Vdash^{\exists} C$. By Lemma 5.4.3 we have that we have that either sequent $S_1 = a \in N(x), a \Vdash^{\exists} A, x : A \preccurlyeq B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma$ or $S_2 = a \in N(x), a \Vdash^{\exists} C, x : A \preccurlyeq B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma$ are derivable (possibly both). However, sequent S_2 is the same sequent as the conclusion of $\mathsf{L} \preccurlyeq$; thus, if we replace the right premiss with this sequent, the application of $\mathsf{L} \preccurlyeq$ would be useless, and we can ignore the case. Thus, replace the right premiss with S_1. Let $\mathcal{D}^-$ be the derivation for S_1: for Lemma 5.4.3 the derivation has height less or equal than $\mathcal{D}_2$, derivation of (2). Moreover, observe that $\mathcal{D}^-$ is a *subderivation* of $\mathcal{D}_2$: it displays the same formulas (and the same rules) except for formula $a \Vdash^{\exists} A$ (and the rules applied to it). Let $\Gamma_1^{S^-} \Rightarrow \Delta_1^{S^-}, \ldots, \Gamma_k^{S^-} \Rightarrow \Delta_k^{S^-}$ be the x-saturated sequents from which $\mathcal{D}^-$ is derived. Each of these sequent is composed of *less* formulas than the x- saturated sequents $\Gamma_1^S \Rightarrow \Delta_1^S, \ldots, \Gamma_k^S \Rightarrow \Delta_k^S$ from which S_2 was derived. This is the reason why we translate x-saturated sequents which are composed of not exactly the same formulas of the original x-saturated sequents, but of a subset of (init) $\left[\overset{\triangledown}{\text{INIT}}\right]^x \rightsquigarrow s(\overset{\triangledown}{\text{INIT}})^x$

(R $\preccurlyeq$) $$\left[\dfrac{\begin{array}{c}\mathcal{D}_1\\ a \in N(x), a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A\end{array}}{\Gamma \Rightarrow \Delta, x : A \preccurlyeq B}\ \mathsf{R}\preccurlyeq\right]^x \rightsquigarrow$$

$$\dfrac{\begin{array}{c}[\mathcal{D}_1]^x\\ s(a \in N(x), a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A)^x\end{array}}{s(\Gamma \Rightarrow \Delta, x : A \preccurlyeq B)^x}\ \mathsf{R}\preccurlyeq^{\mathsf{i}}$$

(L $\preccurlyeq$) $$\left[\dfrac{\begin{array}{c}\mathcal{D}_1\\ (1)\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ (2)\end{array}}{\mathsf{Conc}}\ \mathsf{L}\preccurlyeq\right]^x \rightsquigarrow \dfrac{\begin{array}{c}[\mathcal{D}_1]^x\\ s(1)^x\end{array} \quad \dfrac{\begin{array}{c}[\mathcal{D}^-]^x\\ s(S_1)^x\end{array}}{Q}\ \mathsf{Wk_B}}{s(\mathsf{Conc})^x}\ \mathsf{L}\preccurlyeq^{\mathsf{i}}$$

$(1) = a \in N(x), a \Vdash^{\exists} C, x : A \preccurlyeq B, \mathcal{R}^a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma, a \Vdash^{\exists} B$
$(2) = a \in N(x), a \Vdash^{\exists} A, a \Vdash^{\exists} C, x : A \preccurlyeq B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma$
$\mathsf{Conc} = a \in N(x), a \Vdash^{\exists} C, x : A \preccurlyeq B, \mathcal{R}^a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma$
$S_1 = a \in N(x), a \Vdash^{\exists} A, x : A \preccurlyeq B, \mathcal{R}^a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma$ (from (2) by Lemma 5.4.3)
$Q = A \preccurlyeq B, s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \{\Omega\} \lhd C], [\Sigma, \{\Omega\} \lhd A]$

(Nes) $$\left[\dfrac{\begin{array}{c}\mathcal{D}_1\\ (1)\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ (2)\end{array}}{\mathsf{Conc}}\ \mathsf{Nes}\right]^x \rightsquigarrow \dfrac{\dfrac{\begin{array}{c}[\mathcal{D}_1]^x\\ s(1)^x\end{array}}{P}\ \mathsf{Wk_i} \quad \dfrac{\begin{array}{c}[\mathcal{D}_2]^x\\ s(2)^x\end{array}}{Q}\ \mathsf{Wk_i}}{s(\mathsf{Conc})^x}\ \mathsf{com^i}$$

The underlined formulas are added by $\mathsf{Wk_B}$.
$(1) = a \subseteq b, a \in N(x), b \in N(x), \mathcal{R}^a, \mathcal{R}^b, a \Vdash^{\exists} A, a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma, b \Vdash^{\exists} \Pi$
$(2) = b \subseteq a, a \in N(x), b \in N(x), \mathcal{R}^a, \mathcal{R}^b, a \Vdash^{\exists} A, a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma, b \Vdash^{\exists} \Pi$
$\mathsf{Conc} = a \in N(x), b \in N(x), \mathcal{R}^a, \mathcal{R}^b, a \Vdash^{\exists} A, a \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma, b \Vdash^{\exists} \Pi$
$\mathrm{P} = s(\Gamma)^x \Rightarrow s(\Delta)^x, \left[\Sigma, \Pi, \{\Omega\}, \underline{\{\Xi\}} \lhd A\right], [\Pi, \{\Xi\} \lhd B]$
$\mathrm{Q} = s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \{\Omega\} \lhd A], \left[\Sigma, \Pi, \underline{\{\Omega\}}, \{\Xi\} \lhd B\right]$

Figure 5.4: From **G3V** to $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivations

them. Apply the translation to S_1: $s(S_1)^x = A \preccurlyeq B, s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \{\Omega\} \lhd C]$. Add the missing block to $s(S_1)^x$ by weakening. Application of $\mathsf{L}\preccurlyeq^{\mathsf{i}}$ yields the translation of the conclusion of $\mathsf{L}\preccurlyeq$: $s(\mathsf{Conc})^x = A \preccurlyeq B, s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \{\Omega\} \lhd C]$.

[Nes] The premisses and the conclusion of the rule are the following sequents:

$s(1)^x = s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \Pi, \{\Omega\} \lhd A], [\Pi, \{\Xi\} \lhd B]$;
$s(2)^x = s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \{\Omega\} \lhd A], [\Sigma, \Pi, \{\Xi\} \lhd B]$;
$s(\mathsf{Conc})^x = s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \{\Omega\} \lhd A], [\Pi, \{\Xi\} \lhd B]$.

Sets $\{\Omega\}$ and $\{\Xi\}$ account for the formulas to be added inside blocks, in correspondence with inclusions in $\mathcal{R}^a$ and $\mathcal{R}^b$ (see Definition 5.4.1). If both sets $\mathcal{R}^a$ and $\mathcal{R}^b$ are empty, $\{\Omega\}$ and $\{\Xi\}$ are empty there is no needed to apply $\mathsf{Wk_i}$ to either of the premisses. If

$\mathcal{R}^a$ is not empty, $\{\Omega\}$ is not empty; we need to apply $\mathsf{Wk_i}$ to the second block of $s(2)^x$; Similarly, if $\mathcal{R}^b$ is not empty, $\{\Xi\}$ is not empty; we need to apply $\mathsf{Wk_i}$ to the first block of $s(1)^x$. If both $\{\Omega\}$ and $\{\Xi\}$ are not empty, combine the two above strategies. By means of example, we show the case in which $\mathcal{R}^a = a \subseteq c$ and $\mathcal{R}^b = \emptyset$.

$$\dfrac{a \subseteq b, \dots \Rightarrow \dots \qquad b \subseteq a, \dots \Rightarrow \dots}{a \subseteq c, a, b, a \Vdash^{\exists} A, b \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \Sigma, b \Vdash^{\exists} \Pi}\ \mathsf{Nes}$$

The derivation is translated as follows:

$$\dfrac{s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \Pi, \Omega \lhd A], [\Pi \lhd B], \quad \dfrac{s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \Omega \lhd A], [\Sigma, \Pi \lhd B]}{s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \Omega \lhd A], [\Sigma, \Pi, \Omega \lhd B]}\ \mathsf{Wk_i}}{s(\Gamma)^x \Rightarrow s(\Delta)^x, [\Sigma, \Omega \lhd A], [\Pi \lhd B]}\ \mathsf{com^i}$$

□

We are finally ready to define the full translation for derivations.

Theorem 5.4.5. Let $\mathcal{D}$ be a **G3V** derivation of $\Gamma \Rightarrow \Delta$ in normal form with respect to some x in $\Gamma \cup \Delta$. Then $[\![\mathcal{D}]\!]^x$ is a $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivation of $s(\Gamma \Rightarrow \Delta)^x$.

Proof. By induction on the height of the derivation. Since $\mathcal{D}$ is in normal form with respect to x, it will contain a subderivation $\mathcal{D}^S$ of $\Gamma \Rightarrow \Delta$ from x-saturated sequents $\Gamma^S_1 \Rightarrow \Delta^S_1, \dots, \Gamma^S_n \Rightarrow \Delta^S_n$. Apply translation $[\,]^x$ to $\mathcal{D}^S$, and obtain a derivation of $s(\Gamma \Rightarrow \Delta)^x$ from $s(\Gamma^{S-}_1 \Rightarrow \Delta^{S-}_1)^x, \dots, s(\Gamma^{S-}_n \Rightarrow \Delta^{S-}_n)^x$ (Lemma 5.4.4). Each x-saturated sequent $\Gamma^S_i \Rightarrow \Delta^S_i$ has the form[3]:

$$\mathcal{R}^{a_1}, \dots, \mathcal{R}^{a_n},\ a_1 \in N(x), \dots, a_n \in N(x), a_1 \Vdash^{\exists} A_1, \dots, a_n \Vdash^{\exists} A_n, x : \Gamma^P \Rightarrow$$
$$\Rightarrow x : \Delta^P, a_1 \Vdash^{\exists} \Sigma_1, \dots, a_n \Vdash^{\exists} \Sigma_n.$$

Its translation according to s and x is:

$$s(\Gamma^S_i \Rightarrow \Delta^S_i)^x = \Gamma^P \Rightarrow \Delta^P, [\{\Sigma_1\} \lhd A_1], \dots, [\{\Sigma_n\} \lhd A_n].$$

For each $\Gamma^S_i \Rightarrow \Delta^S_i$, apply the following transformation: go up in the derivation until the x-hypersaturated sequent $\Gamma^H_i \Rightarrow \Delta^H_i$ is reached. The sequent will have the following form:

$$\mathcal{R}^{a_1}, \dots, \mathcal{R}^{a_n}, a_1 \in N(x), \dots, a_n \in N(x), \{y_1 \in a_1\}, \dots, \{y_n \in a_n\}, y_1 : A_1, \dots, y_n : A_n,$$
$$x : \Gamma^P \Rightarrow x : \Delta^P, y_1 : \{\Sigma_1\}, \dots y_n : \{\Sigma_n\}, a_1 \Vdash^{\exists} \Sigma_1, \dots, a_n \Vdash^{\exists} \Sigma_n$$

where $\{y_i \in a_i\}$ is a shorthand for $y_i \in a_i \cup \{y_i \in a_j \,|\, a_i \subseteq a_j \in \mathcal{R}^{a_i}$ *and* $y \in a_i\}$. By Jump lemma either $y_1 : A_1 \Rightarrow y_1 : \{\Sigma_1\}$ is derivable, or this sequent is derivable:

$$\mathcal{R}^{a_1}, \dots, \mathcal{R}^{a_n}, a_1 \in N(x), \dots, a_n \in N(x), \{y_1 \in a_1\}, , \dots, \{y_n \in a_n\}, y_2 : A_2 \dots y_n : A_n,$$
$$x : \Gamma^P \Rightarrow x : \Delta^P, y_2 : \{\Sigma_2\}, \dots y_n : \{\Sigma_n\}, a_1 \Vdash^{\exists} \Sigma_1, \dots, a_n \Vdash^{\exists} \Sigma_n.$$

If $y_1 : A_1 \Rightarrow y_1 : \{\Sigma_1\}$ is not derivable, apply the Jump lemma to the above sequent, and iterate the procedure until a derivable sequent $y_i : A_i \Rightarrow y_i : \{\Sigma_i\}$ is found, for some $1 \leqslant i \leqslant n$. The existence of such a derivable sequent is guaranteed by the Jump lemma, and by the fact that a) the x-hypersaturated sequent is derivable and b) the x-hypersaturated sequent is not derivable in virtue of the part $x : \Gamma^P \Rightarrow x : \Delta^P$. If this

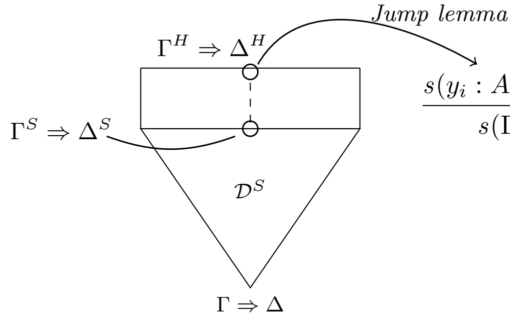
[3]If rule $\mathsf{L} \preccurlyeq$ has been employed in $\mathcal{D}^S$, instead of the x-saturated sequent we choose its "smaller" version $\Gamma^{S-}_i \Rightarrow \Delta^{S-}_i$, since translation s is possibly not applicable to $\Gamma^S_i \Rightarrow \Delta^S_i$. Refer to case $\mathsf{L} \preccurlyeq$ of the proof of Lemma 5.4.4 for details. In any case, the proof strategy remains the same.

was the case, the proof search would have stopped way before, since only propositional rules would have been applied in the derivation.

Suppose $y_i : A_i \Rightarrow y_i : \{\Sigma_i\}$ is derivable; $s(y_i : A_i \Rightarrow y_i : \{\Sigma_i\})^y = A_i \Rightarrow \{\Sigma_i\}$. Application of jump to this sequent yields the translation of the x-saturated sequent

$$s(\Gamma^S \Rightarrow \Delta^S)^x = \Gamma^P \Rightarrow \Delta^P, [\{\Sigma_1\} \lhd A_1], \dots, [\{\Sigma_i\} \lhd A_i] \dots, [\{\Sigma_n\} \lhd A_n].$$

Then, function $[\![\ ]\!]^{y_i}$, with variable y_i as a parameter, has to be recursively invoked to translate the derivation of sequent $y_i : A_i \Rightarrow y_i : \{\Sigma_i\}$, of smaller height than the derivation of $\Gamma \Rightarrow \Delta$. The following picture, with $\Gamma^S \Rightarrow \Delta^S$ x-saturated sequent and $\Gamma^H \Rightarrow \Delta^H$ x-hypersaturated sequent, should help clarify the proof strategy.



$$\dfrac{\dfrac{[\![\ \mathcal{D}]\!]^{y_i}}{s(y_i : A_i \Rightarrow y_i : \{\Sigma_i\})^{y_i}}}{s(\Gamma^S \Rightarrow \Delta^S)^x}\ \mathsf{jump}$$

$\square$

Example 5.4.3. Consider the **G3V** derivation of Example 5.2.1. We show the derivation is translated into a $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivation. As it is, the **G3V** derivation is *not* in normal form with respect to x: we have to saturate its upper part with respect to rules $\mathsf{L} \Vdash^{\exists}$, $\mathsf{R} \Vdash^{\exists}$ and $\mathsf{L} \subseteq$, to the effect that the two upper sequents become two x-hypersaturated sequents. Then, the part of the derivation up to the x-saturated sequents (in this case: premisses of Nesting) is translated employing $[\ ]^x$. Then, we apply $[\![\ ]\!]^x$: consider the left premiss of Nes (x-saturated), and go up to the x-hypersaturated sequent $y \in a, y \in b, z \in a, a \subseteq b, y : B, z : A \Rightarrow y : A, z : B, a \Vdash^{\exists} A, b \Vdash^{\exists} B$. From Lemma 5.4.2, we have that either a) $y : B \Rightarrow y : A, y : B$ is derivable, or b) $y \in a, y \in b, z \in a, z : A \Rightarrow z : B, a \Vdash^{\exists} A, b \Vdash^{\exists} B$ is derivable. Sequent a) is derivable. Thus, we translate sequent $y : B \Rightarrow y : A, y : B$ in the internal sequent calculus, obtaining $B \Rightarrow A, B$. An application of jump allows us to obtain the sequent which is the leftmost application of $\mathsf{com}^{\mathsf{i}}$. A similar reasoning is applied to translate the right premiss of Nes.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\triangledown}{B \Rightarrow A, B}}{\Rightarrow [A, B \lhd B], [B \lhd A]}\ \mathsf{jump} \quad \dfrac{\overset{\triangledown}{A \Rightarrow A, B}}{\Rightarrow [A \lhd B], [A, B \lhd A]}\ \mathsf{jump}}{\Rightarrow [A \lhd B], [B \lhd A]}\ \mathsf{com}^{\mathsf{i}}}{\Rightarrow [A \lhd B], B \preccurlyeq A}\ \mathsf{R} \preccurlyeq^{\mathsf{i}}}{\Rightarrow A \preccurlyeq B, B \preccurlyeq A}\ \mathsf{R} \preccurlyeq^{\mathsf{i}}}{\Rightarrow A \preccurlyeq B \vee B \preccurlyeq A}\ \mathsf{R}\vee$$

5.5 To conclude

The equivalence result presented in this chapter is system-specific: it regards only the basic logic of Lewis' family. Extending the result to other logics is feasible, but not immediate. In particular, defining the direction from the labelled to the internal calculus might require some additional machinery, since each rule needs a specific treatment.

Moreover, it would be interesting to implement the two mappings for $\mathbb{V}$ by means of automated programs. An implementation of the "easy" direction (from internal to labelled calculi) would yield as a result a theorem prover for the labelled calculus **G3V**. A theorem prover for $\mathbb{V}$ was implemented in [36], on the basis of sequent calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$. Other than deciding the validity of a formula in the logic, the theorem prover produces in output a derivation of the formula in $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$. Thus, implementing the translation would yield an automatic procedure to generate a **G3V** derivation from the corresponding $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ derivation. To the best of our knowledge, this would also be the first implementation for a labelled system.

A general study of the complexity of both directions of the translation might be useful. Sequent calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ might yield an optimal decision procedure with respect to logic $\mathbb{V}$, while the complexity of **G3V** is far from optimal. However, the Jump lemma suggests that not all the formulas in a **G3V** derivation are relevant to the final result - thus, some kind of optimization strategy on the labelled calculus might be envisaged.

Other than in these concrete applications, the equivalence result is relevant since it underlines the relationship between the two calculi. As in [40], our translation can be applied between nested sequents and labelled sequents whose underlying structure is that of a tree. Thus, each sequent of the nested calculus encodes the formulas relative to one world of the model - as happens with nested sequents for modal logics. In addition, the translation for $\mathbb{V}$ has to take into account the structure of spheres. Sphere labels are introduced in correspondence with blocks and, in the opposite direction, blocks are composed of formulas labelled with the same neighbourhood label. Thus, the translation assigns a semantic meaning to the elements composing the structure of nested sequents.

In conclusion, Gentzen sequent calculus has to be extended to treat logics with more expressive power than classical logic, such as conditional systems. This can be done either by enriching the language of the calculus or by enriching the structure of sequents. In the case of logic $\mathbb{V}$, the two different strategies amount to the same result: to the sequent calculus is added a syntactic mechanism to encode spheres. For the labelled calculus this is done by means of neighbourhood labels, while the internal calculus uses the block structure.

Chapter 6

Conditional epistemic logics

That's so plausible I can't believe it!
– Turanga Leela, *Futurama*

This final chapter treats the logic of conditional belief: system $\mathbb{CDL}$ (*Conditional Doxastic Logic*), introduced by Board, Baltag and Smets to reason about knowledge and revisable beliefs in a multi-agent setting. More precisely, $\mathbb{CDL}$ is an epistemic logic equipped with an operator of conditional belief, and the system a multi-agent version of Lewis' logic $\mathbb{VTA}$. We apply to $\mathbb{CDL}$ the methods developed in the previous chapters to define proof systems for conditional logic. To this aim, we define a new class of models for the logic: multi-agent neighbourhood models. After showing adequacy of the axiomatization with respect to the semantics, we define the rules of a labelled sequent calculus which internalises the semantic information of neighbourhood models.

6.1 Multi-agent modal epistemic logic

Multi-agent modal epistemic logics have been studied for a long time in formal epistemology, computer science, and notably in artificial intelligence. In this logic, to each agent i is associated a knowledge modality K_i, so that the formula K_iA is read as "the agent i knows A". Through agent-indexed modal operators, epistemic logic can be used to reason about the mutual knowledge of a set of agents. The logic has been further extended by other modalities to encode various types of combined knowledge of agents (e.g. common knowledge). However, knowledge is not the only propositional attitude, and belief is equally significant to reason about epistemic interaction among agents. Board in [16], and then Baltag and Smets in [11, 13, 12] have proposed a logic called $\mathbb{CDL}$ (Conditional Doxastic Logic) for modelling both belief and knowledge in a multi-agent setting. The essential feature of beliefs is that they are *revisable* whenever the agent learns new information. Logic $\mathbb{CDL}$ displays the conditional belief operator $Bel_i(B|A)$, the meaning of which is that agent i believes B if she were to learn A. This operator represents the epistemic reaction of an agent in response to a hypothetical situation: if the agent were to learn A, she would believe that B held in the state of the world *before* the act of learning A. For this reason Baltag and Smets qualify this logic as "static" in contrast to "dynamic" epistemic logic, where the very act of learning (by some form of announcement) may change the agent's beliefs.

The logic $\mathbb{CDL}$ has also been significantly employed in game theory [100]. This logic is suitable to describe game models, i.e., idealized *static* models which represent games. In this setting, the operators of simple belief and knowledge account for a player's doxastic and epistemic attitudes, whereas the conditional belief operator is employed to represent the choices a player maintains as possible at a certain stage, i.e., the strategies a player would apply in response to other player's choices. More generally, the conditional belief operator is suited to represent the states of belief an agent would form in response to an hypothetical situation; thus, $\mathbb{CDL}$ is able to give a complete representation of an agent's epistemic and doxastic attitudes at a given moment of time. The operators of (unconditional) belief and knowledge can be defined in terms of the conditional belief operator:

> $Bel_i B$, "agent i believes B" as $Bel_{ii}(B|\top)$;
> $K_i B$, "agent i knows B" as $Bel_i(\bot|\neg B)$;

the latter meaning that i considers impossible (inconsistent) to learn $\neg B$.

The axiomatization of the operator Bel_i in $\mathbb{CDL}$ internalises the AGM postulates of belief revision [3]. This result is not unexpected, since there is a strong link between conditional logics and epistemic logics. The connection dates back to the work of Gärdenfors, who provided a formalization of the Ramsey's test in terms of belief revision. According to Gärdenfors, conditionals have acceptability conditions (and not truth conditions as in the Stalnaker-Lewis' approach), but he described such conditions by means of a non-probabilistic theory. He defined a semantical theory according to which the meaning assignation of a linguistic entry is not given by considering its relation to the world, but instead by taking into account its relation to a system of beliefs. Then, Gärdenfors defined acceptability conditions for conditionals within this theory, and formalized Ramsey's test on that basis. Gärdenfors theory is not unproblematic[1]; however, to our purpose it suffices to observe that, once extended with relevant epistemic variants, Gärdenfors' theory provides epistemic models for conditional operators.

The semantic interpretation of $\mathbb{CDL}$ is defined in terms of *epistemic plausibility models*. In these models, to each agent i is associated an equivalence relation $\sim_i$, used to interpret knowledge, and a well-founded pre-order $\preceq_i$ on worlds. The relation $\preceq_i$ assesses the relative plausibility of worlds according to an agent i and it is used to interpret conditional beliefs: i believes B conditionally on A in a world x if B holds in *the most plausible worlds* accessible from x in which A holds, the "most plausible worlds" for an agent i being the $\preceq_i$-minimal ones. This semantic approach has been dominant in the studies of $\mathbb{CDL}$; in addition to [16] and [13] we mention [83], [59] and [22].

In this chapter we introduce an alternative semantics for $\mathbb{CDL}$, based on neighbourhood models. The neighbourhood models for $\mathbb{CDL}$ can be seen as a multi-agent generalization of Lewis' spheres models for counterfactual logic $\mathbb{VTA}$. The interpretation of the conditional belief operator Bel_i then coincides with Lewis' semantics of the counterfactual operator in $\mathbb{VTA}$[2]. We believe that neighbourhood models provide a terse interpretation of the epistemic and doxastic modalities, abstracting away from the relational information specified in plausibility models.

After presenting multi-agent neighbourhood models (Section 6.2), we prove adequacy of the axiomatization of $\mathbb{CDL}$ with respect to this class of models (Section 6.3). Then, in Section 6.4 we define **G3CDL**, a labelled sequent calculus for $\mathbb{CDL}$. To the best of our knowledge, **G3CDL** is the first proof system for the logic, which has been studied

[1] Refer to [8] or [78] for a survey on Gärdenfors' account of conditionals. Gärdenfors himself found the so-called Triviality Result in his theory: only trivial models satisfy the Ramsey test with postulates of belief revision.

[2] System $\mathbb{VTA}$, presented in the previous chapter, is conditional logic characterized by the semantic conditions of nesting, total reflexivity and absoluteness.

before only form a semantic viewpoint. The calculus is similar to the labelled systems presented in Chapter 3, as the models for $\mathbb{CDL}$ are a multi-agent version of sphere models for $\mathbb{VTA}$. Section 6.5 proves termination and semantic completeness of the calculus.

Finally, Section 6.6 presents a proof of equivalence between epistemic plausibility models and multi-agent neighbourhood models for $\mathbb{CDL}$, and Section 6.7 extends the multi-agent neighbourhood models to cover other epistemic and doxastic operators, such as safe and strong belief [13, 83], and provides the sequent calculus rules for them.

6.2 Axiomatization and semantics

For i variable ranging over a set of agents $\mathcal{A}$ and p propositional variable, the language of $\mathbb{CDL}$ is defined by means of the following grammar:

$$\mathcal{F}^{\mathbb{CDL}} \ := \ p \mid \bot \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid Bel_i(B|A)$$

where $A, B \in \mathcal{F}^{\mathbb{CDL}}$. In the following, let $\wedge$ and $\vee$ bind stronger than $\rightarrow$ and Bel_i. Negation is defined as $\neg A = A \rightarrow \bot$. The conditional belief operator $Bel_i(B|A)$ is read as "agent i believes B, given A". As mentioned in the previous section, the modalities of unconditional belief and knowledge can be defined in terms of conditional belief as follows:

$Bel_i A =_{def} Bel_i(A|\top)$ (belief)
$K_i A =_{def} Bel_i(\bot|\neg A)$ (knowledge)

In the definition of the operator of conditional belief the "given B" part is to be interpreted as "in case B is added to the set of belief". In other words, B is to be intended as a new belief, and not as a new knowledge, since interpreting B as knowledge would lead to a circularity in the definition $K_i A =_{def} Bel_i(\bot|\neg A)$. An equivalent second-order characterization of knowledge is that $K_i A$ holds if and only if for all B we have $Bel_i(A|B)$, meaning that A will persist as a belief no matter what is learnt.

An axiomatization of $\mathbb{CDL}$ has been discussed in [16], [83] and [13]. We present an alternative axiomatization, $\mathcal{H}_{\mathbb{CDL}}$, equivalent to the one in [13]. The double implication $A \leftrightarrow B$ is defined in the standard way as $(A \rightarrow B) \wedge (B \rightarrow A)$. We denote by $\vdash_{\mathcal{H}_{\mathbb{CDL}}}$ derivability in $\mathcal{H}_{\mathbb{CDL}}$, so $\vdash_{\mathcal{H}_{\mathbb{CDL}}} A$ means that A is a theorem in $\mathcal{H}_{\mathbb{CDL}}$.

(AX.0) Any axiomatization of classical propositional calculus
(AX.1) If $\vdash_{\mathcal{H}_{\mathbb{CDL}}} B$, then $\vdash_{\mathcal{H}_{\mathbb{CDL}}} Bel_i(B|A)$
(AX.2) If $\vdash A \leftrightarrow B$, then $\vdash Bel_i(C|A) \leftrightarrow Bel_i(C|B)$
(AX.3) $(Bel_i(B|A) \wedge Bel_i(B \rightarrow C|A)) \rightarrow Bel_i(C|A)$
(AX.4) $Bel_i(A|A)$
(AX.5) $Bel_i(B|A) \rightarrow (Bel_i(C|A \wedge B) \leftrightarrow Bel_i(C|A))$
(AX.6) $\neg Bel_i(\neg B|A) \rightarrow (Bel_i(C|A \wedge B) \leftrightarrow Bel_i(B \rightarrow C|A))$
(AX.7) $Bel_i(B|A) \rightarrow Bel_i(Bel_i(B|A)|C)$
(AX.8) $\neg Bel_i(B|A) \rightarrow Bel_i(\neg Bel_i(B|A)|C)$
(AX.9) $A \rightarrow \neg Bel_i(\bot|A)$

Figure 6.1: Axiomatization of $\mathbb{CDL}$

Axiom 6 can be equivalently replaced by the following axioms:

(AX.6a) $\neg Bel_i(\neg B|A) \to (Bel_i(C|A) \to Bel_i(C|A \wedge B)$
(AX.6b) $Bel_i(C|A \wedge B) \to Bel_i(B \to C|A)$

Axiom 6b is in turn equivalent to the following axiom:

(AX.10) $(Bel_i(C|A) \wedge Bel_i(C|B)) \to Bel_i(C|A \vee B)$

In terms of belief revision, the above axioms may be understood as a sort of epistemic and internalized version of the AGM postulates. The epistemization rule (2) and distribution axiom (3) express the deductive closure of beliefs. Success axiom (4) ensures that the learned information is included in the set of beliefs. Axioms (5) and (6) encode the *minimal change principle*, a basic assumption of belief revision (see the correspondence with AGM postulates K*7 and K*8). Axiom (7) and (8) express positive and negative introspection of beliefs. Axiom (9), consistency, ensures that learning a true information cannot lead to inconsistent beliefs (it roughly corresponds to AGM K*5). The standard characterization of knowledge as an S5-modality, i.e. the following laws

$$K_i A \to A \qquad K_i A \to K_i K_i A \qquad \neg K_i A \to K_i \neg K_i A$$

can be derived from its definition in terms of conditional belief and the above axioms.

We introduce a semantics for $\mathbb{CDL}$ based on neighbourhood models, or N-models for short. As explained in the previous section, these are a multi-agent version of the *sphere models* introduced by Lewis in [57] for the logic of counterfactuals.

Definition 6.2.1. Let $\mathcal{A}$ be a set of agents. A *multi-agent neighbourhood model* (N-model) has the form $\mathcal{M} = \langle W, \{N\}_{i \in \mathcal{A}}, [\![\ ]\!] \rangle$ where W is a non-empty set; for each $i \in \mathcal{A}$, N_i is a neighbourhood function $N_i : W \to \mathcal{P}(\mathcal{P}(W))$ that assigns a collection of sets of worlds to each world in W; $[\![\ ]\!] : Atm \to \mathcal{P}(W)$ is the propositional evaluation.
For $i \in \mathcal{A}$, $x \in W$, N_i satisfies the following properties:

- *Non-emptiness*: For all $\alpha \in N_i(x)$ it holds that $\alpha \neq \emptyset$;
- *Nesting*: For all $\alpha, \beta \in N_i(x)$ either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$;
- *Total reflexivity*: There exists $\alpha \in N_i(x)$ such that $x \in \alpha$;
- *Local absoluteness*: If $\alpha \in N_i(x)$ and $y \in \alpha$ then $N_i(x) = N_i(y)$;
- *Strong closure under intersection*: If $S \subseteq N_i(x)$ and $S \neq \emptyset$ then $\bigcap S \in S$[3].

The truth conditions for formulas of the language are given inductively by extending the evaluation function $[\![\ ]\!]$ as follows:

- For the Boolean cases the clauses are standard: $[\![A \wedge B]\!] \equiv [\![A]\!] \cap [\![B]\!]$, $[\![A \vee B]\!] \equiv [\![A]\!] \cup [\![B]\!]$, $[\![A \to B]\!] \equiv (W - [\![A]\!]) \cup [\![B]\!]$;
- $x \in [\![Bel_i(B|A)\ ]\!]$ *iff either for all* $\alpha \in N_i(x)$ *it holds that* $\alpha \cap [\![A]\!] = \emptyset$ *or there exists* $\beta \in N_i(x)$ *such that* $\beta \cap [\![A]\!] \neq \emptyset$ *and* $\beta \subseteq [\![A \to B]\!]$.

A formula A is *valid* in $\mathcal{M}$ if $[\![A]\!] = W$. We say that A *is valid in the class of neighbourhood models* if A is valid in every neighbourhood model $\mathcal{M}$.

Observe that total reflexivity entails that every $N_i(x)$ is non-empty, whereas strong closure under intersection always holds in finite models, because of non-emptiness and nesting.
Notational convention: We often write $\mathcal{M}, x \Vdash A$, meaning $x \in [\![A]\!]$. This is further shortened to $x \Vdash A$ whenever $\mathcal{M}$ is unambiguous. Then, we use the local forcing relations introduced in [73]:

[3] The property of strong closure under intersection is needed to prove the equivalence between epistemic plausibility models and multi-agent neighbourhood models. More precisely, this property is needed to ensure the property of *well-foundness* of epistemic plausibility models. Refer to the section 6.6 for the full proof of equivalence.

$\alpha \Vdash^{\forall} A$ *iff for all* $y \in \alpha$ *it holds that* $y \Vdash A$;
$\alpha \Vdash^{\exists} A$ *iff there exists* $y \in \alpha$ *such thaty* $\Vdash A$.

With this notation, the truth condition of conditional belief Bel_i becomes:

$x \Vdash Bel_i(B|A)$ *iff either for all* $\alpha \in N_i(x)$ *it holds that* $\alpha \Vdash^{\forall} \neg A$ *or there exists* $\beta \in N_i(x)$ *such that* $\beta \Vdash^{\exists} A$ *and* $\beta \Vdash^{\forall} A \to B$.

With the notation just introduced, the semantic definitions of the unconditional belief and knowledge can be stated as follows:

$x \Vdash Bel_i B$ *iff there exists* $\beta \in N_i(x)$ *such that* $\beta \Vdash^{\forall} B$;
$x \Vdash K_i B$ *iff for all* $\beta \in N_i(x)$ *it holds that* $\beta \Vdash^{\forall} B$.

6.3 Adequacy of the axiomatization

In this section we prove soundness and completeness of the axiomatization with respect to multi-agent neighbourhood models, N-models. As mentioned in section 6.1, the traditional models defined for $\mathbb{CDL}$ are *epistemic plausibility models*, *EP*-models. The definition of *EP*-models, along with a proof of the equivalence between the class of N-model and *EP*-model for $\mathbb{CDL}$, can be found in Section 6.6.

Theorem 6.3.1. For any formula A, if $\vdash_{\mathcal{H}_{\mathbb{CDL}}} A$, then A is valid in the class of neighbourhood models.

Proof. By induction on the height of the derivation of A defined in the standard way. We show validity of Axiom 6a, Axiom 7 and Axiom 9.
(AX.6a) $\neg Bel_i(\neg B|A) \to (Bel_i(C|A) \to Bel_i(C|A \wedge B)$. Assume that there is a model $\mathcal{M}$ which satisfies the antecedent but does not satisfy the consequent of the axiom at world x. Thus, assume $\mathcal{M}, x \Vdash \neg Bel_i(\neg B|A)$, $\mathcal{M}, x \Vdash Bel_i(C|A)$ and $\mathcal{M}, x \nVdash Bel_i(C|A \wedge B)$. We now have the following:

1. There exists $\alpha \in N_i(x)$ such that $\alpha \Vdash^{\exists} A$
2. For all $\delta \in N_i(x)$, $\delta \Vdash^{\exists} A \;\to\; \delta \Vdash^{\exists} A \wedge B$
3. For all $\alpha \in N_i(x)$ either $\alpha \Vdash^{\forall} \neg A$ or there exists $\beta \in N_i(x)$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to C$
4. There exists $\alpha \in N_i(x)$ such that $\alpha \Vdash^{\exists} A \wedge B$
5. For all $\delta \in N_i(x)$, $\delta \Vdash^{\exists} A \wedge B \;\to\; \delta \Vdash^{\exists} A \wedge B \wedge \neg C$

The first disjunct of 3. does not hold, since it contradicts 1. From the second disjunct of 3, we have that there exists a β_0 such that $\beta_0 \Vdash^{\exists} A$. From 2, we have that $\beta_0 \Vdash^{\exists} A \wedge B$. Then, from 5. we have that $\beta_0 \Vdash^{\exists} A \wedge B \wedge \neg C$. Thus, there exists $y \in \beta_0$ such that $y \Vdash A \wedge B \wedge \neg C$. From 3. we have that $y \Vdash A \to C$: contradiction.
From 3. we also have that $\beta_0 \Vdash^{\forall} A \to C$: contradiction.
(AX.7) $Bel_i(B|A) \to Bel_i(Bel_i(B|A)|C)$. Again, suppose $\mathcal{M}, x \Vdash Bel_i(B|A)$ and $\mathcal{M}, x \nVdash Bel_i(Bel_i(B|A)|C)$. Thus,

1. Either for all $\alpha \in N_i(x)$, $\alpha \Vdash \neg A$ or there exists $\exists \beta \in N_i(x)$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$
2. There exists $\alpha \in N_i(x)$ such that $\alpha \Vdash^{\exists} C$
3. For all $\beta \in N_i(x)$, if $\beta \Vdash^{\exists} C$ then $\beta \Vdash^{\exists} C$ and $\neg Bel_i(B|A)$

From 3. we have

4. There exists $y \in \beta$ such that $y \Vdash C$ and $y \Vdash \neg Bel_i(B|A)$
5. There exists $\gamma \in N_i(y)$ such that $\gamma \Vdash^{\exists} A$ (from 4)

6. for all $\delta \in N_i(y)$, if $\delta \Vdash^{\exists} A$ then $\delta \Vdash^{\exists} A \wedge \neg B$ (from 4)

By the absoluteness condition applied to 4., since $\beta \in N_i(x)$ and $y \in \beta$, we have $N_i(x) = N_i(y)$. Observe that the first disjunct of 1. does not hold, since it contradicts 5. Thus, the second disjunct of 1. holds, and we have that there exists $\beta \in N_i(x) = N_i(y)$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \rightarrow B$. This contradicts with 6.
(AX.9) $A \rightarrow \neg Bel_i(\bot|A)$. Suppose $\mathcal{M}, x \Vdash A$ and $\mathcal{M}, x \Vdash Bel_i(\bot|A)$. Thus, for all $\alpha \in N_i(x)$, $\alpha \Vdash^{\forall} \neg A$ or there exists $\beta \in N_i(x)$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \rightarrow \bot$. By total reflexivity the first disjunct does not hold, since $\mathcal{M}, x \Vdash A$ and there exists $\alpha \in N_i(x)$ such that $x \in \alpha$. The second disjunct is contradictory: we have that there exists $y \in \beta$ such that $y \Vdash A$, and that $y \Vdash A \rightarrow \bot$; thus, $y \Vdash \bot$. □

To prove completeness we have to show the following:

Theorem 6.3.2 (Completeness)**.** For any formula A, if A is valid in the class of neighbourhood models, then $\vdash_{\mathcal{H}_{\mathbb{CDL}}} A$.

We prove the contrapositive: If $\nvdash_{\mathcal{H}_{CDL}} A$, then A is not valid in the class of neighbourhood models. We introduce standard notions and lemmas.

Definition 6.3.1. Given $S \subseteq \mathcal{F}^{\mathbb{CDL}}$, we say that S is *inconsistent* if it has a finite subset $\{B_1, \dots B_n\} \subseteq S$ such that $\vdash_{\mathcal{H}_{CDL}} B_1 \wedge \dots \wedge B_n \rightarrow \bot$. We say that S is *consistent* if it is not inconsistent. We say that $S \subseteq \mathcal{F}^{\mathbb{CDL}}$ is *maximal consistent* if it is consistent and for any formula $A \notin S$, $S \cup \{A\}$ is inconsistent. Let $MAXC(\mathcal{F}^{\mathbb{CDL}})$ denote the set of maximal consistent sets of $\mathcal{F}^{\mathbb{CDL}}$.

Lemma 6.3.3. For $S \subseteq \mathcal{F}^{\mathbb{CDL}}$ consistent set, there exists an $X \in MAXC(\mathcal{F}^{\mathbb{CDL}})$ such that $S \subseteq X$.

Proof. Standard: Let $A_0, A_1, \dots, A_n \dots$ be an enumeration of all formulas of $\mathcal{F}^{\mathbb{CDL}}$. Define a sequence of sets $X_0 = S$, $S_{i+1} = S_i \cup \{A_i\}$ if A_i is consistent with S_i, and $S_{i+1} = S_i$ if not. Then define $X = \bigcup_i X_i$; this set can be proved to be consistent and maximal. □

Lemma 6.3.4. Let X be in $MAXC(\mathcal{F}^{\mathbb{CDL}})$. Then the following properties hold:

(i) For any formula A, either $A \in X$ or $\neg A \in X$
(ii) $A \wedge B \in X$ iff $A \in X$ and $B \in X$
(iii) $A \vee B \in X$ iff $A \in X$ or $B \in X$
(iv) $A \in X$ and $A \rightarrow B \in X$ implies $B \in X$
(v) If $\vdash_{\mathcal{H}_{CDL}} A$ then $A \in X$

The following lemma contains a list of theorems of $\mathbb{CDL}$ used in subsequent proofs.

Lemma 6.3.5. The following are derivable in $\mathbb{CDL}$:

(1) $Bel_i(B|A) \wedge Bel_i(C|A) \rightarrow Bel_i(B \wedge C|A)$
(2) $Bel_i(\bot|A \vee B) \rightarrow (Bel_i(\bot|A) \wedge Bel_i(\bot|B))$
(3) $Bel_i(\bot|A) \rightarrow Bel_i(\neg A|A \vee B)$
(4) If $\vdash_{\mathcal{H}_{CDL}} A \rightarrow B$ then $\vdash_{\mathcal{H}_{CDL}} Bel_i(B|A)$
(5) $Bel_i(\neg D|C \vee D) \rightarrow Bel_i(\neg D|C)$
(6) $Bel_i(D|C) \rightarrow Bel_i(\bot|\neg Bel_i(D|C))$
(7) $\neg Bel_i(D|C) \rightarrow Bel_i(\bot|Bel_i(D|C))$
(8) $(\neg Bel_i(\neg A|A \vee B) \wedge Bel_i(\neg A|A \vee C)) \rightarrow Bel_i(\neg B|B \vee C)$

Proof. For the sake of readability, we use $\vdash$ instead of $\vdash_{\mathcal{H}_{\mathbb{CDL}}}$ to denote derivability in the axiom system.
(1). We have $\vdash B \to (C \to B \wedge C)$, so by Axiom 1, $Bel_i(B \to (C \to B \wedge C)|A)$. By Axiom 3 (twice) and the assumptions we obtain $Bel(B \wedge C|A)$.
(2). It suffices to show that $Bel_i(\perp|A \vee B) \to Bel_i(\perp|A)$. By propositional reasoning, Axiom 1, and Axiom 3, from $Bel_i(\perp|A \vee B)$ follows (a) $Bel_i(A|A \vee B)$. Applying Axiom 5 to (a) and $Bel_i(\perp|A \vee B)$ we get (b) $Bel_i(\perp|A \wedge (A \vee B))$. Since $\vdash A \to A \vee B$, by Axiom 1 we have $Bel_i(A \to A \vee B|A)$; applying Axiom 3 to this formula and to formula $Bel_i(A|A)$ (Axiom 4) we have (c) $Bel_i(A \vee B|A)$. A final application of Axiom 5 to (b) and (c) yields $Bel_i(\perp|A)$.
(3). Applying propositional reasoning, Axiom 1 and Axiom 3 to $Bel_i(\perp|A)$ we get (a) $Bel_i(\neg A|A)$. As in the previous case, we obtain (b) $Bel_i(A \vee B|A)$ from Axiom 1 applied to $\vdash A \to A \vee B$ and Axiom 3. Apply Axiom 5 to (a) and (b) to get $Bel_i(\neg A|A \wedge (A \vee B))$. Since $\vdash A \wedge (A \vee B) \to A \vee B$, by propositional reasoning we have $Bel_i(\neg A|A \vee B)$.
(4). By Axiom 1, $\vdash A \to B$ gives $\vdash Bel_i(A \to B|A)$. By Axiom 4 we also have $\vdash Bel_i(A|A)$, and by Axiom 3 we conclude that $Bel_i(B|A)$.
(5). By Axiom 4 we have $Bel_i(C \vee D|C \vee D)$. By propositional reasoning, we also have (a) $Bel_i(\neg D \to C|C \vee D)$. Apply Axiom 3 to (a) and to the antecedent $Bel_i(\neg D|C \vee D)$ to get (b) $Bel_i(C|C \vee D)$. Then, apply Axiom 5 to the antecedent and (b), and obtain (c) $Bel_i(\neg D|C \wedge (C \vee D))$. Formula $Bel_i(C \vee D|C)$ is derivable, by (4) applied to $\vdash C \to C \vee D$. Apply Axiom 5 again to (c) and (4) and obtain the consequent $Bel_i(\neg D|C)$.
(6). From $Bel_i(D|C)$ and Axiom 7 obtain (a) $Bel_i(Bel_i(D|C)|\neg Bel_i(D|C))$. By Axiom 4 we have (b) $Bel_i(\neg Bel_i(D|C)|\neg Bel_i(D|C))$. Applying (3) of this Lemma to (a) and (b) yields $Bel_i(Bel_i(D|C) \wedge \neg Bel_i(D|C)|\neg Bel_i(D|C))$. This is equivalent to $Bel_i \perp|\neg Bel_i(D|C))$.
(7). From $\neg Bel_i(D|C)$ and Axiom 8 obtain (a) $Bel_i(\neg Bel_i(D|C)|Bel_i(D|C))$. Then, Axiom 4 gives (b) $Bel_i(Bel_i(D|C)|Bel_i(D|C))$. Apply (1) to (a) and (b) and obtain $Bel_i(Bel_i(D|C) \wedge \neg Bel_i(D|C)|Bel_i(D|C)$. Thus, we have that $Bel_i(\perp|Bel_i(D|C))$. (8). We prove the following equivalent formulation: $(Bel_i(\neg A|A \vee C) \wedge \neg Bel_i(\neg B|B \vee C)) \to Bel_i(\neg A|A \vee B)$. First, let us prove the following: $i) Bel_i(\neg A|A \vee C) \to Bel_i(\neg A|A \vee B \vee C)$. It holds that (a) $Bel_i(A \vee B \vee C|A \vee C)$, by (4) and a suitable propositional formula. Apply Axiom 5 to (a) and the antecedent of $i)$ and obtain (b) $Bel_i(\neg A|(A \vee C) \wedge (A \vee B \vee C)$. By Axiom 4 applied to a $\neg A \wedge F$, for an arbitrary formula F, $Bel_i(\neg A|\neg A \wedge F)$. Let $F = \neg C \wedge (A \vee B \vee C)$. Thus we have $Bel_i(\neg A|\neg A \wedge \neg C \wedge (A \vee B \vee C))$, from which by propositional reasoning we have $c)$ $Bel_i(\neg A|\neg(A \vee C) \wedge (A \vee B \vee C))$. From (b), (c) and Axiom 10 we have:

$$(d)\ Bel_i(\neg A|(\neg(A \vee C) \wedge (A \vee B \vee C)) \vee ((A \vee C) \wedge (A \vee B \vee C))$$

By propositional reasoning, this is equivalent to $Bel_i(\neg A|\neg(A \vee C) \vee (A \vee C) \wedge (A \vee B \vee C)$, which is equivalent to $Bel(\neg A|A \vee B \vee C)$.
Then, we prove $ii)$ $\neg Bel_i(\neg B|B \vee C) \to \neg Bel_i(\neg(A \vee B)|A \vee B \vee C)$. We prove the contrapositive: $Bel_i(\neg(A \vee B)|A \vee B \vee C) \to Bel_i(\neg B|B \vee C)$. From the antecedent derive by propositional reasoning (e) $Bel_i(\neg A|A \vee B \vee C)$, and (f) $Bel_i(\neg B|A \vee B \vee C)$. Apply Axiom 5 to the antecedent and (e), and obtain (g) $Bel_i(\neg(A \vee B)|B \vee C)$. Apply Axiom 10 to (f) and (g) to obtain (h) $Bel_i(\neg B|(B \vee C) \wedge (A \vee B \vee C))$. Application of the same axiom to (g) and (h) yields (l) $Bel_i(A \vee B \vee C|B \vee C)$. A final application of Axiom 5 to (h) and (l) yields the desired conclusion $Bel_i(\neg B|B \vee C)$.
We can now proceed with the proof. Apply $i)$ to the first conjunct of the antecedent $Bel_i(\neg A|A \vee C)$ to obtain (a') $Bel_i(\neg A|A \vee B \vee C)$. Apply $ii)$ to the second conjunct of the antecedent $\neg Bel_i(\neg B|B \vee C)$ and obtain (b') $\neg Bel_i(\neg(A \vee B)|A \vee B \vee C)$. Applying

Axiom 6 to (a') and (b') yields $Bel_i(\neg A|(a \vee B \vee C) \wedge (A \vee B))$. Application of the same axiom to this formula and to the derivable formula $Bel_i(A \vee B \vee C|A \vee B)$ yields the desired conclusion $Bel_i(\neg A|A \vee B)$. $\square$

Our goal is to build a canonical neighbourhood model $\mathcal{M}$ such that for any set of formulas S, if S is consistent then it is satisfiable in $\mathcal{M}$. To this regard:

- $W = MAXC(\mathcal{F}^{\mathbb{CDL}})$;
- $[\![p]\!] = \{X \in MAXC(\mathcal{F}^{\mathbb{CDL}}) \mid p \in X\}$, for p atomic formula.

We have to define the neighbourhoods $N_i(X)$ for an element $X \in W$ (and this is the hard part). We proceed similarly as in [57], defining the notion of an 'implausible' set of formulas with respect to X. Then, each implausible set S with respect to X will provide a neighbourhood of X, namely the set of elements of $MAXC(\mathcal{F}^{\mathbb{CDL}})$ which do not contain any formula in S.

Definition 6.3.2. Let $S \subseteq \mathcal{F}^{\mathbb{CDL}}$ and $X \in MAXC(\mathcal{F}^{\mathbb{CDL}})$. Define S to be an *implausible set* with respect to an agent i and a maximal consistent set X whenever the following conditions hold:

(i) For any formula A, if $Bel_i(\perp|A) \in X$ then $A \in S$;
(ii) If $A \in S$ and $B \notin S$ then $Bel_i(\neg A|A \vee B) \in X$.

We denote by $IMPLA_i(X)$ the set of all implausible sets S with respect to X and i.

Intuitively, condition (i) means that S contain all formulas that lead agent i to believe an absurdity, whereas condition (ii) means that for each $A \in S$ and $B \notin S$, agent i considers B strictly more plausible than A, that is, if i learns $A \vee B$ then she would believe $\neg A$, (whence she would believe B, since from $Bel_i(\neg A|A \vee B)$ follows $Bel_i(B|A \vee B)$).

Lemma 6.3.6. The following hold:

(i) If $S_1, S_2 \in IMPLA_i(X)$ then $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$;
(ii) $\mathcal{F}^{\mathbb{CDL}} \in IMPLA_i(X)$;
(iii) Let $S \in IMPLA_i(X)$ with $S \neq \mathcal{F}^{\mathbb{CDL}}$; for any A, if $\vdash_{\mathcal{H}_{CDL}} A$ then $A \notin S$;
(iv) $IMPLA_i(X)$ has a smallest element:

$$S_X^{min} = \{A \in \mathcal{F}^{\mathbb{CDL}} | Bel_i(\perp|A) \in X\}.$$

Proof.

(i) Suppose the contrary and let $A \in S_1 \setminus S_2$ and $B \in S_2 \setminus S_1$; by condition (ii) in Definition 6.3.2 we get $Bel_i(\neg A|A \vee B) \in X$ and $Bel_i(\neg B|A \vee B) \in X$. By (1) of Lemma 6.3.5 we have $Bel_i(\neg(A \vee B)|A \vee B) \in X$, and since $Bel_i(A \vee B|A \vee B) \in X$, we get $Bel_i(\perp|A \vee B) \in X$. By (2) of Lemma 6.3.5, this implies both $Bel_i(\perp|A) \in X$ and $Bel_i(\perp|B) \in X$, violating condition (i) of definition of implausible set for both S_1 and S_2.
(ii) Obvious, since the antecedent of condition (ii) in Definition 6.3.2 is always false.
(iii) Suppose the contrary: let $\vdash_{\mathcal{H}_{CDL}} A$ and $A \in S$. Since $S \neq \mathcal{F}^{\mathbb{CDL}}$, let $B \notin S$. Then by (ii) of Definition 6.3.2 we have (1) $Bel_i(\neg A|A \vee B) \in X$. Since $\vdash_{\mathcal{H}_{CDL}} A$ we also have $A \in X$ and $\vdash Bel_i(\perp|\neg A)$ and therefore $Bel_i(\perp|\neg A) \in X$ so (2) $Bel_i(A|A \vee B) \in X$ by (3) of Lemma 6.3.5. By (1) and (2) $Bel_i(\perp|A \vee B) \in X$, which implies $Bel_i(\perp|A) \in X$, whence we obtain $A \notin X$, thus a contradiction.
(iv) It suffices to show that S_X^{min} satisfies condition (ii) of the definition of implausible set. Let $A \in S_X^{min}$; then $Bel_i(\perp|A) \in X$, whence for any B, by (3) of Lemma 6.3.5, $Bel_i(\neg A|A \vee B) \in X$.

□

For any set $S \subseteq \mathcal{F}^{\mathbb{CDL}}$ we define:

$$CO(S) = \{Y \in MAXC(\mathcal{F}^{\mathbb{CDL}}) \mid Y \cap S = \emptyset\}$$
$$N_i(X) = \{CO(S) \mid S \in IMPLA_i(X) \text{ and } S \neq \mathcal{F}^{\mathbb{CDL}}\}$$

Intuitively, each sphere α will be defined as a set $CO(S)$: a sphere is thus determined by an implausible set of formulas S, i.e. a sphere is the set of worlds not containing any implausible formula with respect to X. Then, $N_i(X)$ is the set of spheres determined by each set of formulas S.

It trivially holds that $CO(\mathcal{F}^{\mathbb{CDL}}) = \emptyset$; furthermore, it can be proved that if $S \in IMPLA_i(X)$ and $S \neq \mathcal{F}^{\mathbb{CDL}}$ then $CO(S) \neq \emptyset$ (the proof is similar to the one of the following Lemma 6.3.7). Observe that the largest neighbourhood is $CO(S_X^{min})$ which contains all Y that do not contain any formula considered "impossible" for X.

The following lemma is similar to Lewis' *Cosphere Lemma* [57], and will be widely used in the sequel.

Lemma 6.3.7. Let $\alpha \in N_i(X)$ with $\alpha = CO(S)$ for some $S \in IMPLA_i(X)$. Then for any formula A it holds that $A \in S$ if and only if for all $Y \in \alpha$ it holds that $A \notin Y$ (thus $\neg A \in Y$).

Proof. To prove direction ($\Rightarrow$), suppose $A \in S$ then by definition of $\alpha = CO(S)$, for all $Y \in \alpha$ it holds $A \notin Y$.

To prove direction ($\Leftarrow$), suppose that for all $Y \in \alpha = CO(S)$ it holds that $A \notin Y$, and by reductio ad absurdum that $A \notin S$. Let us consider the set $\{\neg B \mid B \in S\}$. Suppose first that $\{\neg B \mid B \in S\} \cup \{A\}$ is consistent. Then for some $Z \in MAXC(\mathcal{F}^{\mathbb{CDL}})$, we have $\{\neg B \mid B \in S\} \cup \{A\} \subseteq Z$ (Lemma 6.3.3). We get that $Z \cap S = \emptyset$, so that $Z \in \alpha = CO(S)$. But since $A \in Z$, we have a contradiction with the hypothesis. Thus $\{\neg B \mid B \in S\} \cup \{A\}$ is inconsistent; this means that there is a finite set $\{\neg B_1, \dots, \neg B_n\}$ such that:

$$\vdash_{\mathcal{H}_{CDL}} (\neg B_1 \wedge \dots \wedge \neg B_n) \rightarrow \neg A$$

which is the same as

$$\vdash_{\mathcal{H}_{CDL}} A \rightarrow (B_1 \vee \dots \vee B_n).$$

It follows that

$$(1)\ Bel_i(B_1 \vee \dots \vee B_n | A) \in X$$

For each B_k it holds that $B_k \in S$ and $A \notin S$; thus, by condition (ii) of Definition 6.3.2 we have $Bel_i(\neg B_k | A \vee B_k) \in X$. This implies that $Bel_i(\neg B_k | A) \in X$ for each (i), whence

$$(2)\ Bel_i(\neg(B_1 \vee \dots \vee \neg B_n) | A) \in X$$

But (1) and (2) imply $Bel_i(\perp | A) \in X$. Thus, by condition (i) of Definition 6.3.2, $A \in S$, against the assumption $A \notin S$. □

We are finally ready for the main result. Let us define the canonical model $\mathcal{M} = \langle W, N_i, [\![\ ]\!] \rangle$ as detailed above. We prove that $\mathcal{M}$ is indeed a multi-agent neighbourhood model and that it correctly gives the truth condition for formulas.

The only property we do not show is *strong closure under intersection*, because we do not (yet) know whether this property holds in the canonical model. However, this property is irrelevant for completeness, since the axioms of $\mathbb{CDL}$ are valid in models which do not necessarily satisfy this property, as is shown in the proof of Theorem 6.3.1.

Moreover, by the finite model property (see end of Section 4) it follows that if a formula A is satisfiable in a neighbourhood model then A is satisfiable in a finite model, that in itself satisfies the strong intersection property. Thus the class of formulas which are valid in models that satisfy the strong intersection property is the same as the class of formulas that are valid in models that do not necessarily satisfy this property. No formula can distinguish between models that satisfy and those that do not satisfy the strong intersection property. The situation could be different if we considered *strong* completeness, where we are concerned about derivability of logical consequences of an infinite theory, not just of valid formulas.

Proposition 6.3.8. The model $\mathcal{M} = \langle W, \{N\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$ defined above is a neighbourhood model.

Proof. We show that the properties of nonemptiness, nesting, total reflexivity, and local absoluteness hold in the model.

Non-emptiness: If $\alpha \in N_i(X)$ we want to show that $\alpha \neq \emptyset$. Let $\alpha = CO(S)$ for some $S \in \mathit{IMPLA}_i(X)$ $(S \neq \mathcal{F}^{\mathbb{CDL}})$. Proceed similarly to the $(\Leftarrow)$ direction of Lemma 6.3.7: consider the set $\{\neg B | B \in S\}$, and prove that it is consistent (by contradiction); thus, there is a $Y \in \mathit{MAXC}(\mathcal{F}^{\mathbb{CDL}})$ such that $\{\neg B | B \in S\} \subseteq Y$, from which $Y \in \alpha$.

Nesting: Let $\alpha, \beta \in N_i(X)$. Then for some $S_1, S_2 \in \mathit{IMPLA}_i(X)$, $\alpha = CO(S_1)$ and $\beta = CO(S_2)$. By Lemma 6.3.6, either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. In the former case $\beta \subseteq \alpha$, in the latter $\alpha \subseteq \beta$.

Total reflexivity: Given $X \in W$, let us consider the set $S_X^{min} = \{A \in \mathcal{F}^{\mathbb{CDL}} | \mathit{Bel}_i(\perp|A) \in X\} \in \mathit{IMPLA}_i(X)$. If $A \in S_X^{min}$ then $\mathit{Bel}_i(\perp|A) \in X$; thus by Axiom (9) $\neg A \in X$, whence $A \notin X$. We have shown that $S_X^{min} \cap X = \emptyset$, thus $X \in CO(S_X^{min})$.

Local absoluteness: Let $\alpha \in N_i(X)$ and $Y \in \alpha$; we have to show that $N_i(X) = N_i(Y)$. To this purpose it is enough to show that $\mathit{IMPLA}_i(X) = \mathit{IMPLA}_i(Y)$. To prove this it suffices to show that for any formulas C, D we have $\mathit{Bel}_i(D|C) \in X$ if and only if $\mathit{Bel}_i(D|C) \in Y$, since the conditions (i) and (ii) in Definition 6.3.2 only involve formulas of this form (including the particular case of $D = \perp$). We know that $\alpha \subseteq CO(S_X^{min})$. From $\mathit{Bel}_i(D|C) \in X$, and from (6) of Lemma 6.3.5 follows $\mathit{Bel}_i(\perp|\neg \mathit{Bel}_i(D|C)) \in X$. Thus $\neg \mathit{Bel}_i(D|C)) \in S_X^{min}$, and since $Y \in CO(S_X^{min})$, we have that $\neg \mathit{Bel}_i(D|C) \notin Y$, so $\mathit{Bel}_i(D|C) \in Y$. Conversely, suppose that $\mathit{Bel}_i(D|C) \notin X$, then $\neg \mathit{Bel}_i(D|C) \in X$, thus also $\mathit{Bel}_i(\perp|\mathit{Bel}_i(D|C)) \in X$ (by (7) of Lemma 6.3.5). We have that $\mathit{Bel}_i(D|C) \in CO(S_X^{min})$, and since $Y \in CO(S_X^{min})$ we finally obtain $\mathit{Bel}_i(D|C) \notin Y$. $\square$

Definition 6.3.3. The *weight* of a $\mathbb{CDL}$ formula is defined as follows:

- $\mathtt{w}(p) = \mathtt{w}(\perp) = 1$;
- $\mathtt{w}(A \circ B) = \mathtt{w}(A) + \mathtt{w}(B) + 1$ for $\circ = \{\wedge, \vee, \rightarrow\}$;
- $\mathtt{w}(\mathit{Bel}_i(B|A)) = \mathtt{w}(A) + \mathtt{W}(B) + 2$.

Here is the main proposition.

Proposition 6.3.9. Given the canonical model $\mathcal{M} = \langle W, \{N\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$ defined above, for any formula A and any $X \in W$, we have $X \Vdash A$ if and only if $A \in X$.

Proof. By induction on the weight of A. The base case (A is atomic) holds by definition. The inductive cases of Boolean combinations easily follow by the properties of maximal consistent sets. The only interesting case is the one of $A = \mathit{Bel}_i(D|C)$. We show that $X \Vdash \mathit{Bel}_i(D|C)$ iff $\mathit{Bel}_i(D|C) \in X$.

Suppose that $X \Vdash \mathit{Bel}_i(D|C)$. Thus either (1) for each $\alpha \in N_i(X)$, $\alpha \Vdash^{\forall} \neg C$ or (2) there is there is $\alpha \in N_i(X)$ such that $\alpha \Vdash^{\exists} C$ and $\alpha \Vdash^{\forall} C \rightarrow D$. In case (1), let us consider $\alpha = CO(S_X^{min})$. We have that for all $Y \in \alpha$, $Y \nVdash C$, thus by inductive hypothesis, $C \notin Y$. By Lemma 6.3.7, we get $C \in S_X^{min}$, thus $\mathit{Bel}_i(\perp|C) \in X$, whence also $Bel_i(D|C) \in X$. In case (2), let $\alpha = CO(S)$ for some $S \in IMPLA_i(X)$. Then, since $\alpha \Vdash^{\exists} C$ for some $Y \in \alpha$, we have $Y \Vdash C$; thus by inductive hypothesis, $C \in Y$. By Lemma 6.3.7, $C \notin S$. On the other hand $\alpha \Vdash^{\forall} C \rightarrow D$, that is $\alpha \Vdash^{\forall} \neg(C \wedge \neg D)$, similarly to case (1). Employing Lemma 6.3.7 and the inductive hypothesis, we get that $(C \wedge \neg D) \in S$. Since $C \notin S$, we have that $Bel_i(\neg(C \wedge \neg D)|C \vee (C \wedge \neg D)) \in X$. But this implies that $Bel_i(\neg(C \wedge \neg D)|C) \in X$, that is $Bel_i(C \rightarrow D)|C) \in X$, and finally $Bel_i(D|C) \in X$.

Conversely, suppose that $Bel_i(D|C) \in X$. We distinguish different cases.

Case (1). Suppose that $Bel_i(\perp|C) \in X$. Consider the largest neighbourhood $\alpha = CO(S_X^{min})$. We have that $C \in S_X^{min}$ then for all $Y \in \alpha$ we have $C \notin Y$, so that by inductive hypothesis, $Y \Vdash \neg C$, thus $\alpha \Vdash^{\forall} \neg C$, but this also holds for any other $\beta \in N_i(X)$, since $\beta \subseteq \alpha$. We can conclude that $X \Vdash Bel_i(D|C)$.

Case (2). Suppose that $Bel_i(\perp|C) \notin X$. Subcase (2.1). Suppose that $Bel_i(\perp|C \wedge \neg D) \in X$. Then again consider $\alpha = CO(S_X^{min})$; we have that $C \notin S_X^{min}$, thus by Lemma 6.3.7 for some $Y \in \alpha$, $C \in Y$. By inductive hypothesis $Y \Vdash C$, so that $\alpha \Vdash^{\exists} C$. On the other hand $C \wedge \neg D \in S_X^{min}$ and reasoning as in (case 1), we finally get $\alpha \Vdash^{\forall} C \rightarrow D$. We have shown that $X \Vdash Bel_i(D|C)$.

Subcase (2.2). Suppose that $Bel_i(\perp|C \wedge \neg D) \notin X$. This is the most difficult case. Let us consider the following set:

$$S = \{E \in \mathcal{F}^{\mathbb{CDL}} | \neg Bel_i(\neg(C \wedge \neg D)|(C \wedge \neg D) \vee E) \in X \text{ or } Bel_i(\perp|E) \in X\}$$

We first show that *a)* $C \wedge \neg D \in S$: to see this suppose on the contrary that it does not, then $Bel_i(\neg(C \wedge \neg D)|(C \wedge \neg D)) \in X$. We obtain that $Bel_i(\perp|(C \wedge \neg D)) \in X$, against the hypothesis of subcase (2.2).

We also show that *b)* $C \notin S$. Suppose on the contrary that $C \in S$; since $Bel_i(\perp|C) \notin X$, it must be

$$\neg Bel_i(\neg(C \wedge \neg D)|(C \wedge \neg D) \vee C) \in X$$

But $C \equiv (C \wedge \neg D) \vee C$, thus we have $\neg Bel_i(\neg(C \wedge \neg D)|C) \in X$, that is $\neg Bel_i(C \rightarrow D|C) \in X$, so that finally $\neg Bel_i(D|C) \in X$ against the hypothesis $Bel_i(D|C) \in X$.

We now show that $S \in IMPLA_i(X)$. Clearly S satisfies condition (i) Definition 6.3.2. We want to show that S satisfies also condition (ii). To this purpose let $G \in S$ and $H \notin S$. Since $G \in S$, we have: $Bel_i(\perp|G) \in X$ or $\neg Bel_i(\neg(C \wedge \neg D)|(C \wedge \neg D) \vee G) \in X$. In the former case we get $Bel_i(\neg G|H \vee G) \in X$ by (4) of Lemma 6.3.5, fulfilling condition (ii). Otherwise we have (1) $\neg Bel_i(\neg(C \wedge \neg D)|(C \wedge \neg D) \vee G) \in X$. We have that $H \notin S$, which means that: $Bel_i(\perp|H) \notin X$ and (2) $Bel_i(\neg(C \wedge \neg D)|(C \wedge \neg D) \vee H) \in X$. From (1) and (2) we obtain (by (8) of Lemma 6.3.5) again $Bel_i(\neg G|H \vee G) \in X$. Thus $S \in IMPLA_i(X)$.

Let us consider $\beta = CO(S)$. We have that $C \notin S$ and $C \wedge \neg D \in S$, as shown above in *a)* and *b)*. By Lemma 6.3.7 we have for some $Y \in \beta$, $C \in Y$, whence by inductive hypothesis $Y \Vdash C$ and $\beta \Vdash^{\exists} C$. Similarly by Lemma 6.3.7 for all $Y \in \beta$, $C \wedge \neg D \notin Y$, whence by inductive hypothesis for all $Y \in \beta$ $Y \Vdash \neg(C \wedge \neg D)$, that is $Y \Vdash C \rightarrow D$, that is $\beta \Vdash^{\forall} C \rightarrow D$. We have shown that $X \Vdash Bel_i(D|C)$. □

We conclude the proof of the completeness theorem in the standard way. Suppose that $\nvdash_{\mathcal{H}_{CDL}} A$; then there is $X \in MAXC(\mathcal{F}^{\mathbb{CDL}})$ such that $\neg A \in X$ and $A \notin X$. We consider the canonical model $\mathcal{M} = \langle W, N_i, [\![\ ]\!] \rangle$, we have that $X \in W$ and by the above proposition $X \nVdash A$. Thus A is not valid in $\mathcal{M}$.

6.4 Sequent calculus G3CDL

On the basis of multi-agent neighbourhood models we define a labelled sequent calculus for $\mathbb{CDL}$. The procedure to define the rules follows the methodology established by Negri in [69] of internalizing the possible worlds semantics into the syntax of a contraction-free sequent system. The definition of relational atoms and labelled formulas of the calculus is similar to the one given in Definition 3.3.1 of Chapter 3, with the exception of relational atoms $a \in N(x)$ and formulas $x \Vdash_a A|B$.

Definition 6.4.1. Let $x, y, z, \ldots$ be variables for worlds in a neighbourhood model, and $a, b, c, \ldots$ variables for neighbourhood and $\mathcal{A} = \{i, j, k \ldots\}$ variables for agents. Relational atoms are formulas:

- $a \in N_i(x)$, "neighbourhood a belongs to the set of neighbourhood associated to x";
- $x \in a$, "world x belongs to neighbourhood a";
- $a \subseteq b$, "neighbourhood a belongs to neighbourhood b".

Labelled formulas are defined as follows, for $A \in \mathcal{F}^{\mathbb{CDL}}$:

- Relational atoms are labelled formulas;
- $x : A$, "formula A is true at world x";
- $a \Vdash^{\exists} A$, "A is true at some world belonging to neighbourhood a";
- $a \Vdash^{\forall} A$, "A is true at all worlds belonging to neighbourhood a";
- $x \Vdash_i B|A$, "according to agent i, there exists $\beta \in N(x)$ such that $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$"[4].

The rules of **G3CDL** can be found in Figure 6.2. With $(\mathsf{a!})$ we denote the requirement that label a should not occur in the consequent of the rule. Each semantic condition on neighbourhood models (Definition 6.2.1) is in correspondence with a rule in the calculus. Rule Nes corresponds to the property of nesting in Definition 6.2.1; T corresponds to total reflexivity, and $\mathsf{A_1}$ to local absoluteness. As for non-emptiness, the property is expressed by the rules for local forcing. The property of strong closure under intersection needs not be expressed, because the property holds in finite models and we will prove that the logic has the finite model property (see end of Section 4).

Since we wish to obtain a calculus in which the contraction rule is height-preserving admissible, a few rules keep their principal formula in their premisses: $\mathsf{L}\Vdash^{\forall}$, $\mathsf{R}\Vdash^{\exists}$, $\mathsf{L}>$ and $\mathsf{R}|$. Some extra care is needed for rules that may have instances with a duplication of atomic formulas in their conclusion. In **G3CDL**, the rules which are potentially subject to this condition are Nes and $\mathsf{A_1}$, for the case in which $a = b$. By applying the closure condition, we obtain the following instance of Nes:

$$\frac{a \subseteq a, a \in N_i(x), a \in N_i(x), \Gamma \Rightarrow \Delta \quad a \subseteq a, a \in N_i(x), a \in N_i(x), \Gamma \Rightarrow \Delta}{a \in N_i(x), a \in N_i(x), \Gamma \Rightarrow \Delta}\ \mathsf{Nes}$$

and the contracted instance is

$$\frac{a \subseteq a, a \in N_i(x), \Gamma \Rightarrow \Delta \quad a \subseteq a, a \in N_i(x), \Gamma \Rightarrow \Delta}{a \in N_i(x), \Gamma \Rightarrow \Delta}\ \mathsf{Nes}^*$$

[4] Since the model has the property of nesting of neighbourhood, we omit the requirement $\beta \subseteq \alpha$ from this condition. Refer to Remark 3.3.2 in Chapter 3. Moreover, to simplify the reading in this chapter we write $x \Vdash_i B|A$, with A and B permuted with respect to the definition of Chapter 3. However, the meaning is the same.

Initial sequents

$$\frac{}{x : p, \Gamma \Rightarrow \Delta, x : p}\ \mathsf{init} \qquad \frac{}{\bot, \Gamma \Rightarrow \Delta}\ \bot_{\mathsf{L}}$$

Rules for local forcing

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\ \mathsf{R} \Vdash^{\forall}\ \mathsf{(a!)} \qquad \frac{x : A, x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ \mathsf{L} \Vdash^{\forall}$$

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ \mathsf{R} \Vdash^{\exists} \qquad \frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ \mathsf{L} \Vdash^{\exists}\ \mathsf{(a!)}$$

Propositional rules

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B}\ \mathsf{R}\wedge$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A \vee B}{\Gamma \Rightarrow \Delta, x : A, x : B}\ \mathsf{R}\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \to B, \Gamma \Rightarrow \Delta}\ \mathsf{L}\to \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \to B}\ \mathsf{R}\to$$

Rules for conditional belief

$$\frac{a \in N_i(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A}{\Gamma \Rightarrow \Delta, x : Bel_i(B|A)}\ \mathsf{RB}\ \mathsf{(a!)}$$

$$\frac{a \in N_i(x), x : Bel_i(B|A), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad x \Vdash_i B|A, a \in N_i(x), x : Bel_i(B|A), \Gamma \Rightarrow \Delta}{a \in N_i(x), x : Bel_i(B|A), \Gamma \Rightarrow \Delta}\ \mathsf{LB}$$

$$\frac{a \in N_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\exists} A \quad a \in N_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\forall} A \to B}{a \in N_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A}\ \mathsf{R|}$$

$$\frac{a \in N_i(x), a \Vdash^{\exists} A, a \Vdash^{\forall} A \to B, \Gamma \Rightarrow \Delta}{x \Vdash_i B|A, \Gamma \Rightarrow \Delta}\ \mathsf{L|}\ \mathsf{(a!)}$$

Rules for inclusion

$$\frac{a \subseteq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{Ref} \quad \frac{c \subseteq a, c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}{c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}\ \mathsf{Tr} \quad \frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ \mathsf{L} \subseteq$$

Rules for semantic conditions

$$\frac{a \subseteq b, a \in N_i(x), b \in N_i(x), \Gamma \Rightarrow \Delta \quad b \subseteq a, a \in N_i(x), b \in N_i(x), \Gamma \Rightarrow \Delta}{a \in N_i(x), b \in N_i(x), \Gamma \Rightarrow \Delta}\ \mathsf{Nes}$$

$$\frac{x \in a, a \in N_i(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{T}\ \mathsf{(a!)}$$

$$\frac{a \in N_i(x), y \in a, b \in N_i(x), b \in N_i(y), \Gamma \Rightarrow \Delta}{a \in N_i(x), y \in a, b \in N_i(x), \Gamma \Rightarrow \Delta}\ \mathsf{A} \qquad \frac{a \in N_i(x), y \in a, a \in N_i(y), \Gamma \Rightarrow \Delta}{a \in N_i(x), y \in a, \Gamma \Rightarrow \Delta}\ \mathsf{A}^*$$

Figure 6.2: Rules of **G3CDL**

This rule does not need to be added to the calculus because it reduces to an instance of Ref. As for $\mathsf{A_1}$, the instance with a duplication has the following form:

$$\frac{a \in N_i(x), y \in a, a \in N_i(x), a \in N_i(y), \Gamma \Rightarrow \Delta}{a \in N_i(x), y \in a, a \in N_i(x), \Gamma \Rightarrow \Delta}\ \mathsf{A_1}$$

whereas the contracted instance is rule A^* of Figure 6.2; observe that A^* is not an instance of any of the pre-existing rules of the calculus so it has to be explicitly added in order to satisfy the closure condition and thus ensure admissibility of contraction.

Remark 6.4.1. Sequent calculus **G3CDL** is a multi-agent version of the sequent calculus **G3VTA** presented in Chapter 3. However, for **G3VTA** we defined a total of four rules of absoluteness: A_1 and A_2 in their mono-agent version, plus the contracted instances of the rules.

$$\frac{b \in N(y), a \in N(x), y \in a, b \in N(x), \Gamma, \Rightarrow \Delta}{a \in N(x), y \in a, b \in N(x), \Gamma, \Rightarrow \Delta}\ \mathsf{A}_1 \qquad \frac{b \in N(x), a \in N(x), y \in a, b \in N(y), \Gamma \Rightarrow \Delta}{a \in N(x), y \in a, b \in N(y), \Gamma, \Rightarrow \Delta}\ \mathsf{A}_2$$

For **G3CDL** we define only A (multi-agent), corresponding to A_1, and its contracted instance A^*. The reason is that rule A_2 (multi-agent) of **G3VTA** can be simulated by A and T by means of the following derivation.

$$\cfrac{\cfrac{\cfrac{\cfrac{b \in N(x), a \in N(x), y \in a, b \in N(y), \Gamma \Rightarrow \Delta}{b \in N_i(x), c \in N_i(y), c \in N_i(x), x \in c, a \in N_i(x), y \in a, b \in N_i(y), \Gamma, \Rightarrow \Delta}\ \mathsf{Wk}}{c \in N_i(y), c \in N_i(x), x \in c, a \in N_i(x), y \in a, b \in N_i(y), \Gamma, \Rightarrow \Delta}\ \mathsf{A}}{c \in N_i(x), x \in c, a \in N_i(x), y \in a, b \in N_i(y), \Gamma, \Rightarrow \Delta}\ \mathsf{A}}{a \in N_i(x), y \in a, b \in N_i(y), \Gamma, \Rightarrow \Delta}\ \mathsf{T}$$

Remark 6.4.2. In Chapter 3 we presented different variants of sequent calculus $\mathbb{VTA}$. The rules of sequent calculus **G3CDL** could also have been presented in a slightly different form. We could have adopted $\preccurlyeq_i$ as primitive connective - but we have preferred to have the operator of conditional belief as primitive, as in the original presentations of $\mathbb{CDL}$ [16, 11]. Conversely, the variant for logics with absoluteness of Section 3.8, which avoided references to labelled formulas $a \in N(x)$ cannot be employed to define a sequent calculus for $\mathbb{CDL}$: in the multi-agent formulation we need to keep track of the agent i when introducing a neighbourhood. Thus, formulas $a_i \in N(x)$ need to be explicitly defined.

To prove soundness of **G3CDL** we need to define the notion of realization, which interprets the labelled formulas into the models.

Definition 6.4.2. Let $\mathcal{M} = \langle W, \{N\}_{i \in \mathcal{A}}, [\![\ ]\!] \rangle$ be a neighbourhood model, $\mathcal{S}$ a set of world labels, and $\mathcal{N}$ a set of neighbourhood labels. An $\mathcal{SN}$-realization over $\mathcal{M}$ consists of a pair of functions (ρ, σ) such that

- $\rho : \mathcal{S} \to W$ is a function that assigns to each $x \in \mathcal{S}$ an element $\rho(x)$ of W;
- $\sigma : \mathcal{N} \to \mathcal{P}(W)$ is a function that assigns to each $a \in \mathcal{N}$ an element $\sigma(a)$ of $I(w)$, for some $w \in W$.

Given a sequent $\Gamma \Rightarrow \Delta$, let $\mathcal{S}$ be the set of world labels occurring in $\Gamma \cup \Delta$ and $\mathcal{N}$ ne the set of neighbourhood labels in $\Gamma \cup \Delta$. Let (ρ, σ) be an $\mathcal{SN}$-realization; then, $\Gamma \Rightarrow \Delta$ is satisfiable in $\mathcal{M}$ under (ρ, σ) if the following conditions hold:

- $\mathcal{M} \vDash_{\rho,\sigma} a \in N_i(x)$ if $\sigma(a) \in N_i(\rho(x))$ and $\mathcal{M} \vDash_{\rho,\sigma} a \subseteq b$ if $\sigma(a) \subseteq \sigma(b)$;
- $\mathcal{M} \vDash_{\rho,\sigma} x : A$ if $\rho(x) \Vdash A$;
- $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ if $\sigma(a) \Vdash^{\exists} A$ and $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\forall} A$ if $\sigma(a) \Vdash^{\forall} A$;
- $\mathcal{M} \vDash_{\rho,\sigma} x \Vdash_i B|A$ if for some c $\in N_i(\rho(x))$, $c \Vdash^{\exists} A$ and $c \Vdash^{\forall} A \to B$;
- $\mathcal{M} \vDash_{\rho,\sigma} x \Vdash_i Bel_i(B|A)$ if for all $a \in N_i(\rho(x))$, $a \Vdash^{\forall} A$ or $\mathcal{M} \vDash_{\rho,\sigma} x \Vdash_i B|A$;
- $\mathcal{M} \vDash_{\rho,\sigma} \Gamma \Rightarrow \Delta$ if either $\mathcal{M} \nvDash_{\rho,\sigma} F$ for some formula $F \in \Gamma$ or $\mathcal{M} \vDash_{\rho,\sigma} G$ for some formula $G \in \Delta$.

Then, define $\mathcal{M} \vDash \Gamma \Rightarrow \Delta$ iff $\mathcal{M} \vDash_{\rho,\sigma} \Gamma \Rightarrow \Delta$ for every $\mathcal{SN}$-realization (ρ, σ). A sequent $\Gamma \Rightarrow \Delta$ is said to be *valid* if $\mathcal{M} \vDash \Gamma \Rightarrow \Delta$ holds for every neighbourhood model $\mathcal{M}$, i.e. if $\Gamma \Rightarrow \Delta$ is satisfied for every model $\mathcal{M}$ and for every $\mathcal{SN}$-realization (ρ, σ).

Theorem 6.4.1 (Soundness)**.** If a sequent $\Gamma \Rightarrow \Delta$ is derivable in the calculus, then it is valid in the class of multi-agent neighbourhood models.

Proof. By induction on the height of the derivation of a sequent $\Gamma \Rightarrow \Delta$. If the height of the derivation is 0, the sequent is initial or conclusion of $\bot_{\mathsf{L}}$, and by definition it is valid in the class of multi-agent neighbourhood models. If the height of the derivation is greater than 0, the sequent $\Gamma \Rightarrow \Delta$ has been derived by one of the rules of the calculus **G3CDL**. We prove that all rules preserve validity from the premisses to the conclusion. We show only the cases of RB and LB.

[RB] Suppose the premiss of RB is valid, whereas the conclusion is not. Then there is a model $\mathcal{M}$ and a realization (ρ, σ) which falsify the conclusion, i.e. $\mathcal{M} \vDash_{\rho,\sigma} F$ for all formulas $F \in \Gamma$, $\mathcal{M} \nvDash_{\rho,\sigma} G$ for all formulas $G \in \Delta$ and $\mathcal{M} \nvDash_{\rho,\sigma} x : Bel_i(B|A)$. This means that $\rho(x) \nVdash Bel_i(B|A)$, i.e. there exists a $b \in N_i(\rho(x))$ such that $b \Vdash^{\exists} A$ and for all $c \in N_i(\rho(x))$, $c \Vdash^{\forall} \neg A$ or $c \Vdash^{\exists} \neg(A \to B)$. Now consider the premiss of the rule, and define a new realization (ρ', σ') as follows:

$$\begin{aligned}
&\rho'(x) = \rho(x) \\
&\sigma'(a) = b \\
&\sigma'(t) = \sigma(t), \text{ for } t \neq a
\end{aligned}$$

The realization (ρ', σ') differs from (ρ, σ) only for the interpretation of the neighbourhood label a, which is the new neighbourhood introduced in the premiss. Consider the model $\mathcal{M}$ defined above, and the new realization (ρ', σ'). It holds that $\mathcal{M} \vDash_{\rho',\sigma'} a \in N_i(x)$, $\mathcal{M} \vDash_{\rho',\sigma'} a \Vdash^{\exists} A$, and $\mathcal{M} \vDash_{\rho',\sigma'} F$ for all formulas $F \in \Gamma$, $\mathcal{M} \nvDash_{\rho',\sigma'} G$ for all formulas $G \in \Delta$. Since the premiss of the rule is valid (hypothesis), it holds that $\mathcal{M} \vDash_{\rho',\sigma'} x \Vdash_i B|A$, which means that for some $b \in N_i(\rho(x))$ it holds that $b \Vdash^{\exists} A$ and $b \Vdash^{\exists} A \to B$. However, this is a contradiction with what stated above, i.e. that for all $c \in N_i(\rho(x))$, $c \Vdash^{\forall} \neg A$ or $c \Vdash^{\exists} \neg(A \to B)$. Thus, the conclusion is valid.

[LB] Suppose the premisses of the rule are valid, whereas the conclusion is not. Then, there is be a model $\mathcal{M}$ and a realization (ρ, σ) which falsify the conclusion, i.e. $\mathcal{M} \vDash_{\rho,\sigma} a \in N_i(x)$, $\mathcal{M} \vDash_{\rho,\sigma} x : Bel_i(B|A)$, $\mathcal{M} \vDash_{\rho,\sigma} F$ for all $F \in \Gamma$ and $\mathcal{M} \nvDash_{\rho,\sigma} G$ for all $G \in \Delta$. This means that for some $\sigma(a) \in N_i(\rho(x))$, $\rho(x) \vDash x : Bel_i(B|A)$, i.e. 1) for all $b \in N_i(\rho(x))$ either $b \Vdash^{\forall} \neg A$ or there exists $c \in N_i(x)$ such that $c \Vdash^{\exists} A$ and $c \Vdash^{\forall} A \to B$. Then, since both premisses of the rule are valid (hypothesis) it holds that 2) $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ and 3) $\mathcal{M} \nvDash_{\rho,\sigma} x \Vdash_i B|A$, i.e. 4) for all $c \in N_i(\rho(x))$, $c \Vdash^{\forall} \neg A$ or $c \Vdash^{\exists} \neg(A \to B)$. Now, 2) and 3) cannot be simultaneously satisfied. Suppose 2) holds; then the first term of the disjunction of 1) is not satisfied, and the second term must hold, i.e. there exists $c \in N_i(x)$ such that $c \Vdash^{\exists} A$ and $c \Vdash^{\forall} A \to B$. But these conditions are in contradiction with 4). A similar reasoning applies if 2) holds. Thus, one of the premisses is not valid, against the hypothesis. $\square$

The structural properties of **G3CDL** are the same as the structural properties of **G3VTA**. Labelling formulas with the name of an agent does not have any consequences with respect to proof of admissibility of the structural rules. Thus, the basic definition of weight of a labelled formula, as well as the proofs of admissibility of generalized initial sequents and height-preserving substitutions follow the same pattern as the proofs in Section 3.4 of Chapter 3. The same holds for height preserving admissibility of weakening, height-preserving invertibility of all the rules, height-preserving admissibility of contraction and, most importantly, admissibility of cut.

Theorem 6.4.2. The rule of cut is admissible in **G3CDL**.

From admissibility of cut is possible to prove completeness of **G3CDL** with respect to the axioms of $\mathbb{CDL}$.

Theorem 6.4.3. If a formula A is valid in $\mathbb{CDL}$, then there is a derivation of $\Rightarrow x : A$ in the calculus **G3CDL**.

Proof. We show derivation of a derivation of the left-to-right direction of axiom (6). We omit writing the derivable left premisses of rule RB in $\mathcal{D}$, as well as of rule L $>$.The rules of negation can be defined from the rules of implication, as in Appendix 3.A.

$$\mathcal{D}: \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y : A \cdots \Rightarrow \dots y : A \qquad y : B \cdots \Rightarrow \dots y : B}{y : A, y : B, y \in b, c \in N_i(x), c \Vdash^{\exists} A, b \in N_i(x) \cdots \Rightarrow \dots y : A \wedge B}\,\mathsf{R}\wedge}{y : A, y : B, y \in b, c \in N_i(x), c \Vdash^{\exists} A, b \in N_i(x) \cdots \Rightarrow \dots b \Vdash^{\exists} A \wedge B}\,\mathsf{R}\Vdash^{\exists}}{y \in b, c \in N_i(x), c \Vdash^{\exists} A, b \in N_i(x) \cdots \Rightarrow \dots b \Vdash^{\exists} A \wedge B, y : A \to \neg B}\,\mathsf{R}\to}{c \in N_i(x), c \Vdash^{\exists} A, b \in N_i(x) \cdots \Rightarrow \dots b \Vdash^{\exists} A \wedge B, b \Vdash^{\forall} A \to \neg B}\,\mathsf{R}\Vdash^{\forall}}{c \in N_i(x), c \Vdash^{\exists} A, b \in N_i(x) \cdots \Rightarrow \dots b \Vdash^{\exists} A \wedge B, x \Vdash_i \neg B | A}\,\mathsf{R}|}{b \in N_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \dots x : Bel_i(\neg B|A), b \Vdash^{\exists} A \wedge B}\,\mathsf{RB}$$

$$\mathcal{E}: \quad \dfrac{\dfrac{\dfrac{\dfrac{z : A \cdots \Rightarrow \dots z : A \qquad z : c \cdots \Rightarrow \dots z : C}{z : A \to C, z : A, z : B, z \in b, b \in N_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to C, a \Vdash^{\exists} A \wedge B, \cdots \Rightarrow \dots z : C}\,\mathsf{L}\subseteq}{z : A, z : B, z \in b, b \in N_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \dots z : C}\,\mathsf{L}\Vdash^{\forall}}{z \in b, b \in N_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \dots z : (A \wedge B) \to C}\,\mathsf{R}\to, \mathsf{L}\wedge}{b \in N_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \dots b \Vdash^{\forall} (A \wedge B) \to C}\,\mathsf{R}\Vdash^{\forall}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \mathcal{D}\end{matrix} \qquad \begin{matrix}\vdots\\ \mathcal{E}\end{matrix}}{b \in N_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \to C, a \in N_i(x), a \Vdash^{\exists} A \wedge B, x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x \Vdash_i C|A \wedge B}\,\mathsf{R}|}{x \Vdash_i C|A, a \in N_i(x), a \Vdash^{\exists} A \wedge B, x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x \Vdash_i C|A \wedge B}\,\mathsf{L}|}{a \in N_i(x), a \Vdash^{\exists} A \wedge B, x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x \Vdash_i C|A \wedge B}\,\mathsf{LB}}{x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x : Bel_i(C|A \wedge B)}\,\mathsf{RB}}{x : \neg(Bel_i(\neg B|A)), x : Bel_i(C|A) \Rightarrow x : Bel_i(C|A \wedge B)}\,\mathsf{L}\vee$$

□

Rules for knowledge and simple belief

The modal operators of knowledge and simple belief can be defined in terms of the conditional belief operator: $K_iA = Bel_i(\bot|\neg A)$ and $Bel_iA = Bel_i(A|\top)$. By adopting these definitions, we can add to **G3CDL** the rules displayed below which correspond to the interpretation of these two modalities in neighbourhood semantics.

Rules for knowledge and simple belief

$$\dfrac{a \in N_i(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}{\Gamma \Rightarrow \Delta, x : K_iA}\ \mathsf{LK\ (a!)} \qquad \dfrac{a \in N_i(x), x : K_iA, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{a \in N_i(x), x : K_iA, \Gamma \Rightarrow \Delta}\ \mathsf{RK}$$

$$\dfrac{a \in N_i(x), \Gamma \Rightarrow \Delta, x : Bel_iA, a \Vdash^{\forall} A}{a \in N_i(x), \Gamma \Rightarrow \Delta, x : Bel_iA}\ \mathsf{LSB} \qquad \dfrac{a \in N_i(x), a \Vdash^{\forall} A \Rightarrow \Delta}{x : Bel_iA, \Gamma \Rightarrow \Delta}\ \mathsf{RSB\ (a!)}$$

Figure 6.3: Rules for knowledge and simple belief

These rules are *admissible* in **G3CDL**, i.e. whenever the premiss is derivable, also the conclusion is.By way of example, we show the case of rule RK, where the left premiss of LB is derivable from the premiss of RK and the right premiss of LB is derivable from initial sequents.

$$
\dfrac{
\dfrac{a \in N_i(x), x : Bel_i(\bot|\neg A), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{a \in N_i(x), x : Bel_i(\bot|\neg A), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \neg A}\;\mathsf{R}\neg
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{...y : \neg A \Rightarrow y : \neg A... \quad y : \bot \Rightarrow ...}{y : \neg A \to \bot, y : \neg A, y : \neg A ...\Gamma \Rightarrow \Delta}\;\mathsf{L}\subseteq}{y \in b, y : \neg A, b \Vdash^{\forall} \neg A \to \bot ...\Gamma \Rightarrow \Delta}\;\mathsf{L}\Vdash^{\forall}}{b \in N_i(x), b \Vdash^{\exists} \neg A, b \Vdash^{\forall} \neg A \to \bot ...\Gamma \Rightarrow \Delta}\;\mathsf{L}\Vdash^{\exists}}{x \Vdash_i \bot|\neg A ...\Gamma \Rightarrow \Delta}\;\mathsf{L}|
}{a \in N_i(x), x : Bel_i(\bot|\neg A), \Gamma \Rightarrow \Delta}\;\mathsf{LB}
$$

Example 6.4.1. To illustrate the rules of **G3CDL**, and the difference between the conditional belief operator $Bel_i(B|A)$ and the simple belief operator $Bel_i(A \to B)$, consider the following (modified) example from [100]. Let agent i have the belief that Jones is a coward, formalized as $Bel_i C(j)$. Now, we want to express the fact that if i were to learn that Jones has been sent to battle, $S(j)$, he would no longer believe that he is a coward (since only brave men are sent to battle). If we expressed this fact with the simple belief operator we would end up in a contradiction, because from $\neg Bel_i(S(j) \to C(j))$ we conclude $\neg Bel_i C(j)$. However, if we express it as $\neg Bel_i(C(j)|S(j))$, we do not end up in contradiction, since $\neg Bel_i C(j)$ cannot be derived. We show a derivation of $\neg Bel_i(S(j) \to C(j)) \to \neg Bel_i C(j)$, and an open derivation of $\neg Bel_i(C(j)|S(j)) \to \neg Bel_i C(j)$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bigtriangledown}{a \in N_i(x), y \in a, y : C(j), y : S(j) \Rightarrow x : Bel_i(S(j) \to C(j)), a \Vdash^{\forall} S(j) \to C(j), y : C(j)}}{a \in N_i(x), y \in a, y : C(j), \Rightarrow x : Bel_i(S(j) \to C(j)), a \Vdash^{\forall} S(j) \to C(j), y : S(j) \to C(j)}\;\mathsf{R}\to}{a \in N_i(x), y \in a, y : C(j), \Rightarrow x : Bel_i(S(j) \to C(j)), a \Vdash^{\forall} S(j) \to C(j)}\;\mathsf{R}\Vdash^{\forall}}{a \in N_i(x), a \Vdash^{\exists} C(j), \Rightarrow x : Bel_i(S(j) \to C(j)), a \Vdash^{\forall} S(j) \to C(j)}\;\mathsf{L}\Vdash^{\forall}}{a \in N_i(x), a \Vdash^{\exists} C(j), \Rightarrow x : Bel_i(S(j) \to C(j))}\;\mathsf{LSB}}{Bel_i C(j) \Rightarrow x : Bel_i(S(j) \to C(j)}\;\mathsf{RSB}}{x : \neg Bel_i(S(j) \to C(j)) \Rightarrow x : \neg Bel_i C(j)}\;\mathsf{L}\neg, \mathsf{R}\neg}{\Rightarrow x : \neg Bel_i(S(j) \to C(j)) \to \neg Bel_i C(j)}\;\mathsf{R}\to
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bigtriangledown}{\ldots b \Vdash^{\exists} S(j) \Rightarrow \ldots b \Vdash^{\exists} S(j)} \quad \dfrac{\times}{\ldots b \Vdash^{\exists} S(j) \Rightarrow x \Vdash_i C(j)|S(j), b \Vdash^{\forall} S(j) \to C(j)}}{a \in N_i(x), b \in N_i(x), a \Vdash^{\forall} C(j), b \Vdash^{\exists} S(j) \Rightarrow x \Vdash_i C(j)|S(j)}\;\mathsf{R}|}{a \in N_i(x), a \Vdash^{\forall} C(j) \Rightarrow x : Bel_i(C(j)|S(j))}\;\mathsf{RB}}{x : Bel_i C(j) \Rightarrow x : Bel_i(C(j)|S(j))}\;\mathsf{RSB}}{x : \neg Bel_i(C(j)|S(j)) \Rightarrow x : \neg Bel_i C(j)}\;\mathsf{R}\neg, \mathsf{L}\neg}{\Rightarrow x : \neg Bel_i(C(j)|S(j)) \to \neg Bel_i C(j)}\;\mathsf{R}\to
$$

6.5 Termination and completeness

In this section we show how, adopting a suitable proof search strategy, the calculus yields a decision procedure for $\mathbb{CDL}$. Thus, in the following we consider only derivations whose root formula is a (labelled) formula of $\mathbb{CDL}$. We also prove the completeness of the calculus under the same strategy. The adoption of a proof search strategy is not strictly necessary for completeness, but it ensures that we can extract a finite countermodel from an open or failed derivation branch.

The calculus in itself is not terminating: the rules give rise to the same loops described for **G3VTA**. The strategy to prove termination is the same as the one described in Chapter 3 for **G3CL*** and its extensions: it consists in proving that, under the proof-search strategy, only a finite number of labelled formulas might be generated in root-first proof search. To check that the number of such formulas is finite, we have to check the number of labels - and thus, to check that the acyclic graph of labels generated in the derivation branch is finite. The proof of termination is somehow easier than the general strategy described in Chapter 3: first of all, thanks to the condition of nesting, we no longer need to modify the sequent calculus rules, and the number of neighbourhood labels generated is inferior to the number of labels generated by the corresponding non-nested system (refer to Chapter 3 for details). Furthermore, the structure of labels forms a tree instead of a labelled acyclic graph - the graph was generated by the rules of Replacement, for logics with centering. On the other hand, in **G3CDL** we also have the condition of absoluteness - which we had not considered directly in Chapter 3.

We quickly recall the basic definitions and then provide the full termination proof.

Definition 6.5.1. Let $\mathcal{B} = S_0, S_1, \dots$ with S_i sequent $\Gamma_i \Rightarrow \Delta_i$, for $i = 1, 2, \dots$ and S_0 the sequent $\Rightarrow x : A_0$. Let $\downarrow \Gamma_i / \downarrow \Delta_i$ denote the union of the antecedents / succedents occurring in the branch from S_0 up to S_i. We say a sequent $\Gamma \Rightarrow \Delta$ is saturated if it is not an instance of init and $\bot_{\mathsf{L}}$, if it satisfies the saturation conditions from Definition 3.5.2 from chapter 3 for propositional rules and rules for local forcing. Furthermore, a **G3CDL** sequent satisfies the following:

(RB) If $x : Bel_i(B|A)$ is in $\downarrow \Delta$, then for some a, $a \in N_i(x)$ is in Γ, $a \Vdash^{\exists} A$ is in $\downarrow \Gamma$ and $x \Vdash_i B|A$ is in $\downarrow \Delta$;
(LB) If $a \in N_i(x)$ and $x : Bel_i(B|A)$ are in Γ, then either $a \Vdash^{\exists} A$ is in $\downarrow \Delta$ or $x \Vdash_i B|A$ is in $\downarrow \Gamma$;
(RC) If $a \in N_i(x)$ is in Γ and $x \Vdash_i B|A$ is in Δ, then either $a \Vdash^{\exists} A$ or $a \Vdash^{\forall} A \to B$ are in $\downarrow \Delta$;
(LC) If $x \Vdash_i B|A$ is in $\downarrow \Gamma$, then for some a, $a \in N_i(x)$ is in Γ, $a \Vdash^{\exists} A$ and $a \Vdash^{\forall} A \to B$ are in $\downarrow \Gamma$;
(A) If $a \in N_i(x)$ and $y \in a$ are in Γ, then if $b \in N_i(x)$ is in Γ also $b \in N_i(y)$ is in Γ; if $b \in N_i(y)$ is in Γ also $b \in N_i(x)$ is in Γ.

The definition of the relations $x \to_g a$, $a \to_g x$ and $x \overset{w}{\to} y$ are as in Definition 3.5.4.

Lemma 6.5.1. Given a branch $\mathcal{B}$ as in Definition 6.5.1, the following hold:

(a) The relation $\overset{w}{\to}$ is acyclic and forms a *tree* with root x_0;
(b) All world labels occurring in $\mathcal{B}$ are nodes of the tree, that is letting $\overset{w}{\to}^*$ be the *transitive closure* of $\overset{w}{\to}$, if u occurs in $\downarrow \Gamma_k$, then $x_0 \overset{w}{\to}^* u$.

Proof. Same as proof of Lemma 3.5.3 in Chapter 3. □

We can now define the proof-search strategy. A rule R is said to be *applicable* to a world label x if R is applicable to a labelled formula with label x occurring in the conclusion of a rule. In case of rules A_1 and A_2 of local absoluteness, we say the rules are applied to x (rather than to y).

Definition 6.5.2. When constructing root-first a derivation tree for a sequent $\Rightarrow x_0 : A$, apply the following strategy:

(i) No rule can be applied to an initial sequent;
(ii) If $k(x) < k(y)$ all rules applicable to x are applied before any rule applicable to y.
(iii) Rule T is applied as the first one to each world label x.
(iv) Rules which do not introduce a new label (static rules) are applied *before* the rules which do introduce new labels (dynamic rules), with the exception of T, as in the previous item;

(v) For each x, y and a, static rules A_1 and A_2 are applied before any other static rule;

(vi) A rule R cannot be applied to a sequent $\Gamma_i \Rightarrow \Delta_i$ if $\downarrow\Gamma_i$ and / or $\downarrow\Delta_i$ satisfy the saturation condition associated to R.

It follows from the strategy that if $x \xrightarrow{w} y$, every rule applicable to x is applied before every rule applicable to y. As usual, the size of a formula A, denoted by $|A|$, is the number of symbols that occur in A. The size of a sequent $\Gamma \Rightarrow \Delta$ is the sum of all the sizes of the formulas that occur in it.

Lemma 6.5.2. Given a branch $\mathcal{B}$ as in Definition 6.5.1 and a world label x, we define $\mathsf{Neigh}(x) = \{a \mid x \rightarrow_g a\}$ as the set of neighbourhood labels generated by x, and $\mathsf{World}(x) = \{y \mid x \xrightarrow{w} y\}$ as the set of world labels generated by x. The size of $\mathsf{Neigh}(x)$ and $\mathsf{World}(x)$ is finite, more precisely: $|N(x)| = O(n)$ and $|\mathsf{World}(x)| = O(n^2)$.

Proof. We first prove that $|\mathsf{Neigh}(x)| = O(n)$. By definition, $a \in N(x)$ iff $x \rightarrow_g a$, i.e. if there exists $t \leqslant k$ and there exists $i \in \mathcal{A}$ such that a does not occur in Γ_s for all $s \leqslant t$ and $a \in N_i(x)$ belongs to Γ_t. This means that label a has been introduced either by RB or by $\mathsf{L}|$. Therefore x may create as many neighbourhood labels a as there are formulas $x : Bel_i(B|C)$ occurring in $\downarrow\Gamma_k \cup \downarrow\Delta_k$ (plus one neighbourhood introduced by T) and the number of these formulas is $O(n)$.

We now prove $|\mathsf{World}(x)| = O(n^2)$. By definition $y \in W(x)$ iff $x \xrightarrow{w} y$, i.e. iff for some b it holds that $x \rightarrow_g b$ and $b \rightarrow_g y$. We have just shown that for each x, the number of neighbourhood labels generated by x is $O(n)$. Let us consider $b \rightarrow_g y$. By definition, this means that there exists $t < k$, and there exists an $i \in \mathcal{A}$, such that y does not occur in Γ_s for $s \leqslant t$ and $y \in b$ occurs in Γ_{t+1}. There are several ways in which a formula $y \in b$ can be introduced:

Case 1. The formula $y \in b$ is introduced by a formula $b \Vdash^{\exists} C$ that belongs to $\downarrow\Gamma_k$ by application of rule $\mathsf{L}\Vdash^{\exists}$. There are two subcases, according to how formula $b \Vdash^{\exists} C$ has been derived: (a) $b \Vdash^{\exists} C$ has been introduced by RB applied to a formula $x : Bel_i(D|C)$ that belongs to $\downarrow\Delta_k$ and (b) $b \Vdash^{\exists} C$ has been introduced by $\mathsf{L}|$ applied to a formula $x \Vdash_i D|C$ that belongs to $\downarrow\Gamma_k$. In turn, this formula has been introduced by LB applied to a formula $x : Bel_i(D|C)$ that belongs to $\downarrow\Gamma_k$. In case (a), we notice again that RB can be applied only *once* to each formula $x : Bel_i(D|C)$ that occurs in the consequent, and it generates exactly *one* new neighbourhood label b and one formula $b \Vdash^{\exists} C$. Similarly in case (b) $\mathsf{L}>$ can be applied only *once* to $x \Vdash_i D|C$ and generates *one* new neighbourhood label b and *one* formula $b \Vdash^{\exists} C$. By the saturation condition, each formula $x \Vdash_i D|C$ in turn is introduced by (LB) applied only *once* to one formula $x : Bel_i(D|C)$ that occurs in $\downarrow\Gamma_k$. Now each rule $L\Vdash^{\exists}$ generates exactly *one* new world label for each $b \Vdash^{\exists} C$ that occurs in $\downarrow\Gamma_k$ and, as we have just shown the number of such formulas is bounded by the number of formulas of type $x : Bel_i(D|C)$ that occur in $\downarrow\Gamma_k$, and this number is is $O(n)$. Therefore we can conclude that the number of new world labels introduced in this case is $O(n)$.

Case 2. The formula $y \in b$ is introduced by a formula $b \Vdash^{\forall} C$ that belongs to $\downarrow\Delta_k$ by application of rule $\mathsf{R}\Vdash^{\forall}$. But a formula $b \Vdash^{\forall} C$ may be introduced only by an application of $\mathsf{R}|$ to a formula $u \Vdash_i F|E$, where $C = E \rightarrow F \in \downarrow\Delta_k$. In turn, a formula of type $u \Vdash_i F|E$ may be introduced only by an application of RB. Let us consider the set S_b of formulas C such that $S_b = \{C \mid b \Vdash^{\forall} C \text{ belongs to } \downarrow\Delta_k\}$. It holds that:

$$\begin{aligned} S_b &= \{C \mid b \Vdash^{\forall} C \text{ belongs to } \downarrow\Delta_k\} \\ &= \{E \rightarrow F \mid \exists u \exists i \,.\, u \Vdash_i F|E \text{ belongs to } \downarrow\Delta_k\} \\ &= \{E \rightarrow F \mid \exists u \exists i \,.\, u : Bel_i(F|E) \text{ belongs to } \downarrow\Delta_k\} \end{aligned}$$

The cardinality of S_b is the same as the cardinality of the set $\{E \rightarrow F \mid \exists u \exists i \,.\, u : Bel_i(F|E) \text{ belongs to } \downarrow\Delta_k\}$; thus, for each $b \in W(x)$, $|S_b| = O(n)$. In the present case, each $b \in W(x)$ generates $O(n)$ labels.
Then, since $|\mathsf{Neigh}(x)| = O(n)$ we finally get that $|\mathsf{World}(x)| = O(n^2)$. $\square$

Proposition 6.5.3. Any derivation branch $\mathcal{B} = \Gamma_0 \Rightarrow \Delta_0 \ , ..., \Gamma_k \Rightarrow \Delta_k, \Gamma_{k+1} \Rightarrow \Delta_{k+1}, ...$ of a derivation starting from $\Gamma_0 \Rightarrow \Delta_0 \equiv \Rightarrow x_0 : A_0$ built in accordance with the strategy is finite.

Proof. Let us consider a branch $\mathcal{B}$, and suppose by contradiction that $\mathcal{B}$ is not finite. Let $\Gamma^* = \bigcup_k \Gamma_k$ and $\Delta^* = \bigcup_k \Delta_k$; then, Γ^* is infinite. All labelled formulas in Γ^* are subformulas of A_0; however, the subformulas of A_0 are finitely many (namely they are $O(n)$, where n is the length of A_0); thus Γ^* must contain infinitely many labels. By Lemma 6.5.2, Γ^* must contain infinitely many *world* labels, since each world label x generates only $O(n)$ neighbourhood labels. Let us consider now the tree determined by the relation $\stackrel{w}{\rightarrow}^*$ with root x_0. By Lemma 6.5.1, each label in any Γ_k occurs in the tree, which therefore is *infinite*. By Lemma 6.5.2, every label in the tree has $O(n^2)$ successors, thus a finite number. By König's lemma, the tree must contain an *infinite path*: $x_0 \stackrel{w}{\rightarrow} x_1 \stackrel{w}{\rightarrow} \ldots \stackrel{w}{\rightarrow} x_t \stackrel{w}{\rightarrow} x_{t+1} \ldots$, with all x_t being different. We observe that (a) infinitely many x_t must be generated by dynamic rules applied to subformulas of A_0, but (b) these formulas are finitely many, thus there must be a subformula of A_0 which is used infinitely many times to "generate" world labels (or better to generate a neighbourhood label from which a further world label is generated).

There are two cases: this subformula is of type $Bel_i(D|C)$ and occurs in Δ^* or it is of type $\Vdash_i B|A$ and occurs in Γ^* (in this latter case it is not properly a subformulas of A_0 but it is derived form a subformula of A_0).

In the first case, for some x_t we have that $x_t : Bel_i(D|C)$ occurs in some $\Delta_{s(x_t)}$; furthermore, for some a such that $k(a) = s(x_t) + 1$, we have that $a \in N_i(x_t), a \Vdash^{\exists} C \in \Gamma_{s(x_t)+1}$ and $x_t \Vdash_i D|C \in \Delta_{s(x_t)+1}$. Moreover, we have $a \rightarrow_g x_{t+1}$. There must be in the sequence an x_r with $r > t$, such that $x_r : Bel_i(D|C)$ occurs in some $\Delta_{s(x_r)}$ and for a new b, that is with $k(b) = s(x_r) + 1$, we have that $(*)$ $b \in N_i(x_r), b \Vdash^{\exists} C$ belongs to $\Gamma_{s(x_r)+1}$, $x_r \Vdash_i D|C$ occurs in $\Delta_{s(x_r)+1}$ and $b \rightarrow_g x_{t+1}$. By the definition of the strategy, we have that $a \in N_i(x_r)$, thus a itself fulfils the saturation condition for (RB) applied to $x_r : Bel_i(D|C)$ belongs to $\Delta_{s(x_r)}$. Thus, step $(*)$ violates the strategy and we get a contradiction.

The second case displays a similar situation: for some t, $x_t \Vdash_i D|C$ occurs in some $\Gamma_{s(x_t)}$ and for a new a, with $k(a) = s(x_t) + 1$, we have that $a \in N_i(x_t), a \Vdash^{\exists} C$ occurs in $\Gamma_{s(x_t)+1}$ and $a \Vdash^{\forall} C \rightarrow D$ occurs in $\Gamma_{s(x_t)+1}$. Moreover, we have that $a \rightarrow_g x_{t+1}$. Similarly there must be an x_r in the sequence with $r > t$, such that $x_r \Vdash_i D|C$ occurs in some $\Gamma_{s(x_r)}$ and for a new b, with $k(b) = s(x_r) + 1$, we have that we have that $(**)$ $b \in N_i(x_r), b \Vdash^{\exists} C$ occurs in $\Gamma_{s(x_r)+1}$ and $b \Vdash^{\forall} C \rightarrow D$ occurs in $\Gamma_{s(x_r)+1}$. By definition of the strategy we have that $a \in N_i(x_r)$, thus a itself fulfils the saturation condition for $\mathsf{L}|$ applied to $x_r \Vdash_i D|C$ occurring in $\Gamma_{s(x_r)}$. Step $(**)$ violates the strategy, and we get a contradiction. ☐

The previous proof actually shows something stronger than termination of each derivation branch. The proof demonstrates that a formula of type $Bel_i(B|A)$ or $x \Vdash_i B|A$ cannot be used twice to generate two world labels that occur in the same path of the label tree associated to the derivation. Therefore, given an initial formula A_0, the number of formulas of type $Bel_i(B|A)$ or $x \Vdash_i B|A$ that can be generated in the derivation of $\Rightarrow x : A_0$ is bounded by $O(n)$, with n length of A_0. As a consequence, we have the following:

Remark 6.5.1. The height of each branch of a derivation defined as described in Proposition 6.5.3 is is bounded by $O(n)$; thus, the height of the derivation is bounded by $O(n)$, where n is the length of A_0.

Termination of proof search under the strategy is now an obvious consequence:

Theorem 6.5.4 (Termination)**.** Proof search built in accordance with the strategy for any sequent of the form $\Rightarrow x_0 : A_0$ always comes to an end after a finite number of steps. More precisely, the maximal size of each sequent is $O(n^{4n+2})$, and the maximal length of a derivation branch is bounded by $O(n^{2n+1} \cdot n^{4n+2}) = O(n^{6n+3})$.
Furthermore, each sequent that occurs as a leaf of the derivation tree is either an initial sequent or a saturated sequent.

Proof. Consider a branch of a derivation tree whose root is the sequent $\Rightarrow x_0 : A_0$, and build the *finite* tree structure with all the labels that occur in the derivation. The root of the tree will be the label x_0, and all the other labels that occur in $\downarrow \Gamma_k$ will occur as nodes in the tree. As above, $n = |A_0|$.

By Proposition 6.5.1 we have that the height of the label tree associated with the derivation is bounded by $O(n)$.

Then, by Lemma 6.5.2 we have that the number of world labels and of neighbourhood labels that can be generated from each node is finite, and it is bounded by n^2, i.e. it is $O(n^2)$.

Let us consider a derivation tree with root $\Rightarrow x_0 : A_0$. The number of world labels that occur in each branch $\downarrow \Gamma_k$ is bounded by $O(n^{2n})$. The number of neighbourhood labels occurring in $\downarrow \Gamma_k$ is bounded by the number of world labels multiplied by the maximal number of labels generated by each world, that is at most n. Thus, the number of neighbourhood labels is bounded by $O(n^{2n} \cdot n) = O(n^{2n+1})$.

The maximal size of each sequent occurring in the derivation is given by the maximal number of labelled formulas multiplied by the maximal number of subformulas of A_0, which is bounded by n: thus, $O(n^{2n+1} \cdot n) = O(n^{2n+2})$. However, this measure is not sufficient, since it takes into account only formulas of the form $x : F$, $a \Vdash^Q F$ or $x \Vdash_i F|G$. We have to calculate also the number of formulas of the form $y \in b$ and $b \in N_i(x)$ which could have been introduced in the derivation by $\mathsf{L}\subseteq$ or T. The cardinality of the set $\{(y \in b) \mid y, b \text{ occurs in } \downarrow \Gamma_k\}$ is given by $n^{2n+1} \cdot n^{2n+1} = n^{4n+2}$. Thus, the maximal size of the sequents is bounded by $O(n^{4n+2})$.

Finally, the maximal length of each derivation branch is calculated by taking into account the maximal size of the sequents and the maximal number of rules which can be applied to it. We have to distinguish between rules which can be applied more than once (rules $\mathsf{L}\Vdash^{\forall}$, $\mathsf{R}\Vdash^{\exists}$, $\mathsf{R}|$ and LB) and rules which can be applied only once (all the others). The rules which can be applied more than once can be applied as many times as the number of labels occurring in the sequent, i.e. $O(n^{2n+1})$. Thus, the maximal length of a derivation branch is bounded by $O(n^{2n+1} \cdot n^{4n+2}) = O(n^{6n+3})$.

To prove the second part of the theorem, consider a branch $\Gamma_0 \Rightarrow \Delta_0, ..., \Gamma_n \Rightarrow \Delta_n$. As we have just proved, every branch of a derivation tree is finite. The leaf of the branch will be the sequent $\Gamma_n \Rightarrow \Delta_n$, and no rule is applicable to it; thus, trivially, the sequent is either an initial sequent or it is saturated. □

From the proof of Theorem 4.5 we have the following:

Proposition 6.5.5. The validity of a formula A in $\mathbb{CDL}$ can be decided in *NEXPTIME*.

We know that multi-agent $S5$ is a fragment of $\mathbb{CDL}$. By the complexity result in [28] we immediately obtain that *PSPACE* is the lower bound for deciding validity of a $\mathbb{CDL}$ formula. We conjecture that *PSPACE* is also the upper bound for the logic; this problem will be considered in further research.

We show that the calculus is complete under the terminating strategy of Definition 6.5.2. The model construction is similar to the one given in Section 3.6.

Theorem 6.5.6. Let $\Gamma \Rightarrow \Delta$ be the upper sequent of a saturated branch $\mathcal{B}$ in a derivation tree. Then there exists a finite countermodel $\mathcal{M}$ to $\Gamma \Rightarrow \Delta$.

Proof. Let $\Gamma \Rightarrow \Delta$ be the upper sequent of a saturated branch $\mathcal{B}$. By theorem 6.5.4, $\mathcal{B}$ is finite. We construct a model $\mathcal{M}_{\mathcal{B}}$ and a realization (ρ, σ), and show that the model satisfies all formulas in $\downarrow \Gamma$ and falsifies all formulas in $\downarrow \Delta$. Let

$$S_{\mathcal{B}} = \{x \,|\, x \ \in (\downarrow \Gamma \cup \downarrow \Delta)\} \text{ and } N_{\mathcal{B}} = \{a \,|\, a \in (\downarrow \Gamma \cup \downarrow \Delta)\}.$$

Then, associate to each $a \in N_{\mathcal{B}}$ a neighbourhood α_a, such that $\alpha_a = \{y \in S_{\mathcal{B}} \,|\, y \in a \text{ belongs to } \Gamma\}$, thus $\alpha_a \subseteq S_{\mathcal{B}}$. We define a neighbourhood model $\mathcal{M}_{\mathcal{B}} = \langle W, \{N\}_{i \in \mathcal{A}}, [\![\ ]\!]\rangle$ as

- $W = S_{\mathcal{B}}$, i.e. the set W consists of all the labels occurring in the saturated branch $\mathcal{B}$;
- For each $x \in W$, $N_i(x) = \{\alpha_a \,|\, a \in N_i(x) \text{ belongs to } \downarrow \Gamma\}$;
- For p atomic, $[\![p]\!] = \{x \in W \,|\, x : p \text{ belongs to } \downarrow \Gamma\}$.

We first show that:

$(*)$ If $a \subseteq b$ belongs to Γ, then $\alpha_a \subseteq \alpha_b$.

To this aim, suppose $y \in \alpha_a$. This means that $y \in a$ belongs to Γ; then, by the saturation condition $L \subseteq$ also $y \in b$ belongs to Γ. By definition of the model we have $y \in \alpha_b$, and thus that $\alpha_a \subseteq \alpha_b$.
We now show that $\mathcal{M}_{\mathcal{B}} = \langle W, \{N\}_{i \in \mathcal{A}}, [\![\ ]\!]\rangle$ satisfies the properties of a multi-agent neighbourhood model, namely non-emptiness (trivial), total reflexivity, nesting and local absoluteness. Strong closure under intersection follows from finiteness, cf. the end of this section.

Total reflexivity: According to the saturation condition T, for every x that occurs in $\downarrow \Gamma \cup \downarrow \Delta$ also $a \in N_i(x)$, $x \in a$ occur in Γ; then, by definition of $\mathcal{M}_{\mathcal{B}}$, $\alpha_a \in N_i(x)$ and $x \in \alpha_a$.

Nesting: Suppose $\alpha_a \in N_i(x)$ and $\alpha_b \in N_i(x)$. We want to to show that $\alpha_a \subseteq \alpha_b$ or $\alpha_b \subseteq \alpha_a$. By definition of the model, from $\alpha_a \in N_i(x)$ and $\alpha_b \in N_i(x)$ it follows that $a \in N_i(x)$ and $b \in N_i(x)$ both belong to Γ. From the saturation condition S, we have that $a \subseteq b$ or $b \subseteq a$ belong to Γ and we conclude by the fact $(*)$ above.

Local absoluteness: Suppose $\alpha_a \in N_i(x)$ and $y \in \alpha_a$. We want to show that $N_i(x) = N_i(y)$. Suppose $\alpha_b \in N_i(x)$; by definition of the model we have that $a \in N_i(x)$, $y \in a$ and $b \in N_i(x)$ all belong to Γ. By the saturation condition A, also $b \in N_i(y)$ belongs to Γ; thus, by definition, $\alpha_b \in N_i(y)$ holds. For the opposite inclusion apply the same reasoning, exploiting the second condition of the saturation condition A.

Next, define a realization (ρ, σ) such that $\rho(x) = x$ and $\sigma(a) = \alpha_a$. We now prove the following, where $\mathcal{F}$ denotes any formula of the language, i.e. $\mathcal{F}$ is $a \in N_i(x)$, $x \in A$, $a \subseteq b$, $x \Vdash^{\forall} A$, $x \Vdash^{\exists} A$, $x \Vdash_i B|A$, $x : A$, $x : Bel_i(B|A)$:

[Claim 1] If $\mathcal{F}$ is in $\downarrow \Gamma$, then $\mathcal{M}_{\mathcal{B}} \models \mathcal{F}$;
[Claim 2] If $\mathcal{F}$ is in $\downarrow \Delta$, then $\mathcal{M}_{\mathcal{B}} \not\models \mathcal{F}$;

The two claims are proved by cases, by induction on the weight of the formula $\mathcal{F}$. Cases of A relational atom, atomic formula, propositional formula and formula $a \Vdash^{\exists} B$ and $\Vdash^{\forall} B$ are similar to the corresponding cases in proof of Theorem 3.6.1 in Chapter 3. We only show the cases of $A \equiv x \Vdash_i B|A$ and $x : Bel_i(B|A)$.

If $x \Vdash_i B|A$ is in $\downarrow \Gamma$, then by the saturation condition LC for some i, a it holds that $a \in N_i(x)$ is in Γ, and $a \Vdash^{\exists} A$, $a \Vdash^{\forall} A \to B$ are in $\downarrow \Gamma$. By inductive hypothesis, $\mathcal{M}_{\mathcal{B}} \models \alpha_a \Vdash^{\exists} A$, and $\mathcal{M}_{\mathcal{B}} \models \alpha_a \Vdash^{\forall} A \to B$. By definition, this yields $\mathcal{M}_{\mathcal{B}} \models x \Vdash_i B|A$.

If $x \Vdash_i B|A$ is in $\downarrow \Delta$, consider an arbitrary neighbourhood γ_c in $N_i(x)$. Then by definition of $\mathcal{M}_\mathcal{B}$ we have that $c \in N_i(x)$ is in Γ; apply the saturation condition RC and obtain that either $c \Vdash^\exists A$ or $c \Vdash^\forall A \to B$ is in $\downarrow \Delta$. By inductive hypothesis, either $\mathcal{M} \nvDash \gamma_c \Vdash^\exists A$ or $\mathcal{M}_\mathcal{B} \nvDash \gamma_c \Vdash^\forall A \to B$. In both cases, by definition $\mathcal{M}_\mathcal{B} \nvDash x \Vdash_i B|A$.

If $x : Bel_i(B|A)$ is in $\downarrow \Gamma$, then it is also in Γ. Consider an arbitrary neighbourhood α_a in $N_i(x)$. By definition of $\mathcal{M}_\mathcal{B}$ we have that $a \in N_i(x)$ is in Γ; apply the saturation condition LB and conclude that either $a \Vdash^\exists A$ is in $\downarrow \Delta$, or $x \Vdash_i B|A$ is in $\downarrow \Gamma$. By inductive hypothesis, it holds that either $\mathcal{M}_\mathcal{B} \nvDash \alpha_a \Vdash^\exists A$ or $\mathcal{M}_\mathcal{B} \vDash x \Vdash_i B|A$. In both cases, by definition $\mathcal{M}_\mathcal{B} \vDash x : Bel_i(B|A)$.
If $x : Bel_i(B|A)$ is in $\downarrow \Delta$, by the saturation condition RB for some i, a it holds that $a \in N_i(x)$ is in Γ, $a \Vdash^\exists A$ is in $\downarrow \Gamma$ and $x \Vdash_i B|A$ is in $\downarrow \Delta$. By inductive hypothesis, $\mathcal{M}_\mathcal{B} \vDash \alpha_a \Vdash^\exists A$ and $\mathcal{M}_\mathcal{B} \nvDash x \Vdash_i B|A$, thus, by definition, we have $\mathcal{M}_\mathcal{B} \nvDash x : Bel_i(B|A)$. □

The completeness of the calculus is an obvious consequence:

Theorem 6.5.7 (Completeness)**.** If A is valid then it is derivable in **G3CDL**.

Theorem 4.8.2 together with the soundness of **G3CDL** provides a constructive proof of *the finite model property* of $\mathbb{CDL}$: if A is satisfiable in a model (i.e. $\neg A$ is not valid), then, by the soundness of **G3CDL** $\neg A$ is not provable, thus by Theorem 4.8.2 we can build a finite countermodel that falsifies $\neg A$, i.e. which satisfies A.

6.6 Relating the old and the new

In this section we recall the semantics of plausibility models, an earlier semantics for $\mathbb{CDL}$ described in the literature. We relate this semantics to the neighbourhood semantics we have formerly introduced and prove that the two systems are equivalent, i.e. that they validate exactly the same formulas. Observe that this result provides an alternative (indirect) proof of soundness and completeness of the axiomatization of $\mathbb{CDL}$ with respect to plausibility models.

Epistemic plausibility models

Epistemic plausibility models are versatile structures that have been used in a variety of different contexts by logicians, game theorists, and computer scientists, as emphasised in the recent survey article by Pacuit in [83]. Epistemic plausibility models, here called *EP*-models for short, also come with different names depending on the context of inquiry: Board, for instance, calls them *Belief Revision Structures* [16].

Epistemic plausibility models are Kripke structures that display for each agent both an equivalence relation over worlds, defining knowledge (as in standard epistemic models) and a plausibility relation, which is used to define beliefs. The intuition is that an agent's beliefs are the propositions that hold in the worlds (state of affairs, scenarios) that the agent considers the most plausible.

We recall a few preliminary notions. A *pre-order* $\preceq$ over a set W is a reflexive and transitive relation over W. Given $S \subseteq W$, $\preceq$ is *connected* over S if for all $x, y \in S$ either $x \preceq y$ or $y \preceq x$. An *infinite descending* $\preceq$*-chain* over W is a sequence $\{x_n\}_{n \geq 0}$ of elements of W such that for all n, $x_{n+1} \preceq x_n$ but $x_n \npreceq x_{n+1}$. We say that $\preceq$ is *well-founded* over W if there are no infinite descending $\preceq$-chains over W. Given $S \subseteq W$, let $Min_\preceq(S) \equiv \{u \in S \,|\, \forall z \in S \,.\, z \preceq u \to u \preceq z\}$. Observe that whenever $\preceq$ is connected over S the definition $Min_\preceq(S)$ can be simplified to $Min_\preceq(S) = \{u \in S \,|\, \forall z \in S \,.\, u \preceq z\}$. Finally, the well-foundedness property can be equivalently stated as: for each $S \subseteq W$ if $S \neq \emptyset$ then $Min_\preceq(S) \neq \emptyset$.

Definition 6.6.1. Let $\mathcal{A}$ be a set of agents; an *epistemic plausibility model*

$$\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$$

consists of a nonempty set W of elements called "worlds"or "states"; for each $i \in \mathcal{A}$, an equivalence relation $\sim_i$ over W (with $[x]_{\sim_i} \equiv \{w \mid w \sim_i x\}$); for each $i \in \mathcal{A}$, a well-founded pre-order $\preceq_i$ over W; a valuation function $[\![\ ]\!] : Atm \to \mathcal{P}(W)$. The preorder $\preceq_i$ satisfies the following properties:

- *Plausibility implies possibility*: If $w \preceq_i v$ then $w \sim_i v$.
- *Local connectedness*: If $w \sim_i v$ then $w \preceq_i v$ or $v \preceq_i w$ (in other words, $\preceq_i$ is connected over every equivalence class of $\sim_i$).

The truth conditions for Boolean combinations of formulas are the standard ones; the truth condition for the conditional belief operator is the following:

$$[\![Bel_i(B|A)]\!] \equiv \{x \in W \mid Min_{\preceq_i}([x]_{\sim_i} \cap [\![A]\!]) \subseteq [\![B]\!]\}$$

A formula A is *valid* in a model $\mathcal{M}$ if $[\![A]\!] = W$ and that A is *valid in the class of epistemic plausibility models* if A is valid in every epistemic plausibility model.

The following proposition, proved by unfolding the definitions, gives an equivalent formulation of the truth condition of the conditional operator Bel_i given in Section 2.2. From now on, we shall use this formulation.

Proposition 6.6.1. Given any *EP*-model $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$, with $x \in W$ we have that $\mathcal{M}, x \Vdash Bel_i(B|A)$ iff:

Either for all $y \sim_i x$, $y \Vdash \neg A$ or there exists $y \sim_i x$ such that $y \Vdash A$ and for all $z \preceq_i y$, $z \Vdash A \to B$

Proof. *(Only if)* Assume $\mathcal{M}, x \Vdash Bel_i(B|A)$, that is, $Min_{\preceq_i}([x]_{\sim_i} \cap [\![A]\!]) \subseteq [\![B]\!]$. Now, it is either true or false that for all y, $y \sim_i x$ implies $y \Vdash \neg A$: if it is true, we immediately get the result. Else, for some y, $y \sim_i x$ and $y \Vdash A$. Hence $S_A \equiv Min_{\preceq_i}(\{w \mid w \sim_i x, w \Vdash A\}) \neq \emptyset$ from the well-foundedness of $\preceq_i$. Given any $z \in S_A$, given any world y such that $y \sim_i x$ and $y \Vdash A$, we have $z \preceq_i y$ since $\preceq_i$ is a total preordering. Hence $z \Vdash B$ from our initial assumption, so that $z \Vdash A \to B$.

(If) Assume that for all y, $y \sim_i x$ implies $y \Vdash \neg A$ or there is $y \sim_i x$ such that $y \Vdash A$ and for all z, $z \preceq_i y$ implies $z \Vdash A \to B$. If the first disjunct holds, then S_A is empty, which makes the *(If)*-direction trivially true. If the second disjunct holds, then there is some y with $y \sim_i x$ such that $y \Vdash A$ (i.e. S_A is nonempty) and for all z, $z \preceq_i y$ implies $z \Vdash A \to B$. Let $w \in S_A$. We then have $w \preceq_i y$ and therefore $w \Vdash A \to B$. Since $w \in S_A$, we also have $w \Vdash A$, so that $w \Vdash B$ follows, hence the claim $Min_{\preceq_i}([x]_{\sim_i} \cap [\![A]\!]) \subseteq [\![B]\!]$. □

Observation 6.6.1. Recall the definitions of the operators of unconditional belief and knowledge in terms of the conditional belief operator: $Bel_i A =_{def} Bel_i(A|\top)$ and $K_i A =_{def} Bel_i(\bot|\neg A)$. The truth conditions for these operators in plausibility models are the following:

$$[\![Bel_i A]\!] \equiv \{x \in W \mid Min_{\preceq_i}([x]_{\sim_i}) \subseteq [\![A]\!]\}$$
$$[\![K_i A]\!] \equiv \{x \in W \mid [x]_{\sim_i} \subseteq [\![A]\!])\}$$

By Proposition 6.6.1 it is possible to reformulate the above conditions as follows:

$\mathcal{M}, x \Vdash Bel_i A$ *iff* *there exists $y \sim_i x$ such that $y \Vdash A$ and for all $z \preceq_i y$, $z \Vdash A$*;
$\mathcal{M}, x \Vdash K_i(A)$ *iff* *for all $y \sim_i x$, $y \Vdash A$.*

Equivalence between models

We now show the equivalence between neighbourhood models, here called N-models, and epistemic plausibility models, here called *EP*-models. The proof relies on the basic correspondence between partial orders and topologies from [7]; thus, it is similar to the equivalence between neighbourhood models and plausibility models defined in Chapter 3. However, the present result is adapted to the setting of multi-agent neighbourhood models. To prove these results, we need to define a suitable measure of weight for $\mathbb{CDL}$ formulas. The definition of weight of $\mathbb{CDL}$ formulas is given in Definition 6.3.3.

Theorem 6.6.2. If a formula A is valid in the class *EP*-models, then it is valid in the class of multi-agent N-models.

Proof. Given a N-model $\mathcal{M}_N$ we build an *EP*-model $\mathcal{M}_P$ and we show that for any formula A, if A is valid in $\mathcal{M}_P$ then A is valid in $\mathcal{M}_N$. Let $\mathcal{M}_N = \langle W, \{N\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$ be a multi-agent N-model. We construct a *EP*-model $\mathcal{M}_P = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$, by stipulating:

- $x \sim_i y$ iff there exists $\alpha \in N_i(x)$ such that $y \in \alpha$
- $x \preceq_i y$ for all $\alpha \in N_i(y)$, if $y \in \alpha$ then $x \in \alpha$.

We first show that $\sim_i$ is an equivalence relation.

- *Reflexivity.* By total reflexivity there exists $\alpha \in N_i(x)$ such that $x \in \alpha$ holds, thus $x \sim_i x$.
- *Symmetry.* Suppose $x \sim_i y$, this means that there exists $\alpha \in N_i(x)$ such that $y \in \alpha$; by local absoluteness we get $N_i(x) = N_i(y)$. By total reflexivity, there exists $\beta \in N_i(x)$ such that $x \in \beta$, thus also $\beta \in N_i(y)$, and this shows $y \sim_i x$.
- *Transitivity.* Suppose $x \sim_i y$ and $y \sim_i z$, i.e., there exist $\alpha \in N_i(x)$ such that $y \in \alpha$ and there exists $\beta \in N_i(y)$ such that $z \in \beta$; by local absoluteness of N_i we have $N_i(x) = N_i(y)$; therefore there exists $\beta \in N_i(x)$ such that $z \in \beta$, which means $x \sim_i z$.

Next we prove that $\preceq_i$ such as constructed satisfies reflexivity, transitivity, plausibility implies possibility, local connectedness, and well-foundedness:

- *Reflexivity.* Trivial since for all $\alpha \in N_i(x)$ it holds that if $x \in \alpha$ then $x \in \alpha$.
- *Transitivity.* Suppose $x \preceq_i y$ and $y \preceq_i z$, we have 1) for all $\alpha \in N_i(y)$ if $y \in \alpha$ then $x \in \alpha$ and 2) for all $\beta \in N_i(z)$ if $z \in \beta$ then $y \in \beta$. Let $z \in \beta$. Then, from 2) we have $y \in \beta$ and from 1) $x \in \beta$ follows, i.e. for all $\beta \in N_i(z)$ if $z \in \alpha$ then $x \in \beta$ holds. This means $x \preceq_i z$.
- *Local connectedness*: by contradiction suppose that $x \sim_i y$ holds, but that neither $x \preceq_i y$ nor $y \preceq_i x$ holds. By definition of $\preceq_i$ we have:

  for some $\beta \in N_i(y)$, $y \in \beta$ and $x \notin \beta$
  for some $\gamma \in N_i(x)$, $x \in \gamma$ and $y \notin \gamma$.

  Since $x \sim_i y$, by reflexivity there exists $\alpha \in N_i(x)$ such that $y \in \alpha$, whence by local absoluteness $N_i(y) = N_i(x)$. Thus both $\beta, \gamma \in N_i(x)$ and by nesting $\beta \subseteq \gamma$ or $\gamma \subseteq \beta$ holds. If the former holds we get $y \in \gamma$, if the latter holds $x \in \beta$, in both cases reaching a contradiction.
- *Plausibility implies possibility.* Suppose $x \preceq_i y$; by definition, it holds that for all $\alpha \in N_i(y)$ if $y \in \alpha$ then $x \in \alpha$. By total reflexivity, there exists $\beta \in N_i(y)$ such that $y \in \beta$, thus we get $x \in \beta$. Therefore we have that there exists $\beta \in N_i(y)$ such that $x \in \beta$, which means $y \sim_i x$, whence $x \sim_i y$ by symmetry.
- *Well-foundedness.* If $\mathcal{M}_N$ is finite there is nothing to prove. Suppose then that $\mathcal{M}_N$ is *infinite*. Suppose by contradiction that there is an infinite descending chain $\{z_k\}_{k\geq 0}$, i.e. such that for all k:

$z_{k+1} \preceq_i z_k$ and $z_k \not\preceq_i z_{k+1}$

Observe that by definition of $\preceq_i$, plausibility implies possibility, and local absoluteness we obtain that for all $k, h \geq 0$, it holds that $N_i(z_k) = N_i(z_h) = \ldots = N_i(z_0)$. Thus by definition of $\preceq_i$, since for all $k \geq 0 \,.\, z_k \not\preceq_i z_{k+1}$, we get that for all $z_k \in \{z_k\}_{k\geq 0}$ there exists $\beta_{z_{k+1}} \in N_i(z_0)$ such that:

$(*)$ $\quad z_{k+1} \in \beta_{z_{k+1}}$ and $z_k \notin \beta_{z_{k+1}}$.

Consider the set $T = \{\beta_{z_{k+1}} \mid z_k \in \{z_k\}_{k\geq 0}\}$. T is nonempty; thus by strong closure under intersection it follows that that $\bigcap T \in T$, and also $\bigcap T \neq \emptyset$. Obviously, we have that

$(**)$ $\quad$ for all $\beta \in T$, $\bigcap T \subseteq \beta$.

Since $\bigcap T \in T$, we have $\bigcap T = \beta_{z_{t+1}}$ for some $z_t \in \{z_k\}_{k\geq 0}$. By using $(*)$ *twice* (namely for z_{t+1} and for z_{t+2}) we have $z_{t+1} \in \beta_{z_{t+1}}$ and $z_{t+1} \notin \beta_{z_{t+2}}$, thus $\bigcap T = \beta_{z_{t+1}} \not\subseteq \beta_{z_{t+2}}$ against $(**)$.

We now prove that for any $x \in W$ and formula A

$$(a)\ \mathcal{M}_N, x \Vdash A \text{ iff } \mathcal{M}_P, x \Vdash A$$

We proceed by induction on the weight of A. The base case (A atomic) holds by definition, as $[\![\ ]\!]$ is the same in the two models. For the propositional cases statement (a) easily follows by inductive hypothesis. We only consider the case $A = Bel_i(C|B)$. To simplify the notation we write $u \Vdash_P B$ instead of $\mathcal{M}_P, u \Vdash B$ and $u \Vdash_N B$ instead of $\mathcal{M}_N, u \Vdash B$.

[$\Rightarrow$] Suppose that $x \Vdash_N Bel_i(C|B)$. This means that:

Either for all $\alpha \in N_i(x)$ it holds that $\alpha \Vdash^{\forall} \neg B$ or there exists $\beta \in N_i(x)$ such that $\beta \Vdash^{\exists}$ and $\beta \Vdash^{\forall} B \to C$.

We consider the two cases separately. Suppose first that for all $\alpha \in N_i(x)$, $\alpha \Vdash^{\forall} \neg B$ holds; we show that for all y, $y \sim_i x$ implies $y \Vdash_P \neg B$. Let $y \sim_i x$; then, by definition, there exists $\alpha \in N_i(x)$ such that $y \in \alpha$; since $\alpha \Vdash^{\forall} \neg B$ we get $y \Vdash_N \neg B$, and thus by inductive hypothesis $y \Vdash_P \neg B$ holds.

Suppose now that there exists $\beta \in N_i(x)$ such that $\beta \Vdash^{\exists} B$ and $\beta \Vdash^{\forall} B \to C$ hold. We prove that there exists w such that $w \sim_i x$ and $w \Vdash_P B$, and that for all z, if $z \preceq_i w$ then $z \Vdash_P B \to C$. The hypothesis gives in particular that there exists $\beta \in N_i(x)$ such that $\beta \Vdash^{\exists} B$, whence there exists $w \in \beta w$ such that $w \Vdash_N B$. Thus, $x \sim_i w$ and by inductive hypothesis also $w \Vdash_P B$. Now let $z \preceq_i w$. By definition this means that for all $\gamma \in N_i(w)$, if $w \in \gamma$ then $z \in \gamma$. Therefore, since $w \in \beta$, also $z \in \beta$. From $\beta \Vdash^{\forall} B \to C$ we get $z \Vdash_N B \to C$, whence also $z \Vdash_P B \to C$ by inductive hypothesis.

[$\Leftarrow$] Suppose that $x \Vdash_P Bel_i(C|B)$ holds. This means that:

Either for all $y \sim_i x$, $y \Vdash_P \neg B$ or there exists a $w \sim_i x$ and $w \Vdash_P B$ and for all $z \preceq_i w$, $z \Vdash_P B \to C$.

As above, there are two cases to consider. Suppose first that for all $y \sim_i x$ it holds that $y \Vdash_P \neg B$. Let $\alpha \in N_i(x)$ and $u \in \alpha$. By definition $u \sim_i x$, thus by hypothesis $u \Vdash_P \neg B$ and by inductive hypothesis $u \Vdash_N \neg B$. This means that $\alpha \Vdash^{\forall} \neg B$ (first case of truth definition of Bel_i in neighbourhood models).

Suppose now that there exists w such that $w \sim_i x$ and $w \Vdash_P B$ and for all z $z \preceq_i w$ implies $z \Vdash_P B \to C$. From $w \sim_i x$ (hypothesis) it follows by definition that there exists $\alpha \in N_i(x)$ such that $w \in \alpha$. By local absoluteness, $N_i(x) = N_i(w)$. Now consider the set $S = \{\beta \in N_i(x) \mid w \in \beta\}$. It holds that $\alpha \in S$, and that $S \neq \emptyset$. Let $\gamma = \bigcap S$. By strong closure under intersection, $\gamma \in S \subseteq N_i(x)$, thus $\gamma \in N_i(x)$. But $w \in \gamma$ and since we have $w \Vdash_P B$, we also have $w \Vdash_N B$ by inductive hypothesis. We have obtained that $\gamma \Vdash^{\exists} B$. We still have to prove that $\gamma \Vdash^{\forall} B \to C$. Let $u \in \gamma$; let us prove that $u \preceq_i w$ and $u \Vdash_N B \to C$. We first show that $u \preceq_i w$. To this purpose, let $\delta \in N_i(w)$ with $w \in \delta$ (by definition of $\preceq_i$), and we prove that $u \in \delta$: since $N_i(x) = N_i(w)$, also $\delta \in N_i(x)$, whence $\delta \in S$ and $\gamma \subseteq \delta$; therefore $u \in \delta$, and $u \preceq_i w$. Since $u \preceq_i w$ by hypothesis we have $u \Vdash_P B \to C$ and by inductive hypothesis $u \Vdash_N B \to C$. Thus, $\gamma \Vdash^{\forall} B \to C$.

(End of the proof). Suppose that A is valid in $\mathcal{M}_P$. Thus for all $w \in W$, we have $w \Vdash_P A$, and by (a) we have also $w \Vdash_N A$ for all $w \in W$, which means that A is valid in $\mathcal{M}_N$. So we proved that if A is valid in $\mathcal{M}_P$ then A is also valid in $\mathcal{M}_N$. Finally, given a N-model $\mathcal{M}_N$, we build an *EP*-model $\mathcal{M}_P$ as above. By the proof given above, if A is valid in $\mathcal{M}_P$, A is valid in $\mathcal{M}_N$. □

Theorem 6.6.3. If a formula A is valid in the class of multi-agent N-models, then it is valid in the class of *EP*-models.

Proof. Given a *EP*-model $\mathcal{M}_P$ we build an N-model $\mathcal{M}_N$ and we show that for any formula A, if A is valid in $\mathcal{M}_N$ then A is valid in $\mathcal{M}_P$. The result easily follows from this fact.
Let $\mathcal{M}_P = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$ be an *EP*-model. We build a N-model $\mathcal{M}_N$ as follows. Let $u \in W$, and define its downward closed set $\downarrow^{\preceq_i} u$ according to $\preceq_i$ as $\downarrow^{\preceq_i} u = \{v \in W \mid v \preceq_i u\}$ Now we define the model $\mathcal{M}_N = \langle W, \{N\}_{i\in\mathcal{A}}, [\![\ ]\!]\ \rangle$, where for any $x \in W$

$$N_i(x) = \{\downarrow^{\preceq_i} u \mid u \sim_i x\}$$

We first show that $\mathcal{M}_N$ is indeed a N-model.

- *Non-emptiness:* Let $\alpha \in N_i(x)$, then $\alpha = \downarrow^{\preceq_i} u$ for some $u \sim_i x$ and since $u \in \downarrow^{\preceq_i} u$, we have $\alpha \neq \emptyset$.
- *Nesting:* Let $\alpha, \beta \in N_i(x)$. Then, $\alpha = \downarrow^{\preceq_i} u$ for some $u \sim_i x$ and $\beta = \downarrow^{\preceq_i} v$ for some $v \sim_i x$. We can conclude $u \sim_i v$, and by local connectedness we have $u \preceq_i v$ or $v \preceq_i u$. It is immediate to see that this entails $\downarrow^{\preceq_i} u \subseteq \downarrow^{\preceq_i} v$ or $\downarrow^{\preceq_i} v \subseteq \downarrow^{\preceq_i} u$, from which the result follows.
- *Total reflexivity:* Obvious since $x \in \downarrow^{\preceq_i} x$.
- *Local absoluteness:* We first prove the following fact: if $y \sim_i x$ then $N_i(y) = N_i(x)$. Let $y \sim_i x$ and $\downarrow^{\preceq_i} z \in N_i(y)$, then $z \sim_i y$, so that by transitivity $z \sim_i x$, thus $\downarrow^{\preceq_i} z \in N_i(x)$ and hence $N_i(y) \subseteq N_i(x)$. The opposite inclusion $N_i(x) \subseteq N_i(y)$ is proved in the same way. As for local absoluteness: suppose $\alpha \in N_i(x)$ and $y \in \alpha$. This means that $\alpha = \downarrow^{\preceq_i} u$ for some $u \sim_i x$; since $y \in \downarrow^{\preceq_i} u$, we have $y \preceq_i u$ and by plausibility implies possibility $y \sim_i u$ and therefore also $y \sim_i x$. Then we apply the above fact.
- *Closure under intersection:* In the finite case, this property immediately follows from properties non-emptiness and nesting. If $\mathcal{M}_P$ is infinite, let $S \subseteq N_i(x)$, $S \neq \emptyset$, with S countable so that $S = \{\alpha_h \mid h \geqslant 0\}$ where $\alpha_h = \downarrow^{\preceq_i} x_h$ for $x_h \sim_i x$. We prove that

  $(*)$ there exists $\alpha_h \in S$ such that $\forall \alpha_k \in S \,.\, \alpha_h \subseteq \alpha_k$

  If $(*)$ holds then $\alpha_h = \bigcap S$ and $\alpha_h \in S$ and the proof is over. Suppose by contradiction that $(*)$ does not hold. This means that: 1) for all $\alpha_h \in S$ there exists $\exists \alpha_k \in S$ such that $\alpha_h \nsubseteq \alpha_k$. Thus, by the property of spheres nesting 2) for all $\alpha_h \in S$ there exists $\alpha_k \in S$ such that $\alpha_k \subset \alpha_h$. From 2), by denumerable dependent choice, we can build an infinite (strictly decreasing) chain of neighbourhoods

$\alpha_1 \to \alpha_2 \to \alpha_3 \to \ldots$

For every $n \geq 1$ we have by definition that $\alpha_n = \downarrow^{\preceq_i} u_n$. Let $v_n \in \alpha_n - \alpha_{n+1}$, $v_{n+1} \in \alpha_{n+1} - \alpha_{n+2}$, etc. We have $v_{n+1} \preceq_i u_{n+1}$ by construction and it is enough to prove that $u_{n+1} \preceq_i v_n$ to conclude by transitivity that $v_{n+1} \preceq_i v_n$. By construction, we have $v_n \not\preceq_i u_{n+1}$ and therefore by local connectedness, $u_{n+1} \preceq_i v_n$. Moreover by $v_n \not\preceq_i u_{n+1}$ it also follows that $v_n \not\preceq_i v_{n+1}$. We have thus an infinitely descending $\preceq_i$-chain of worlds $\{v_n\}_{n\geq 1}$, against the assumption of well-foundedness of W. We reached a contradiction from the negation of $(*)$; therefore, $(*)$ holds.

We now prove that for any $x \in W$ and formula A

$$(b)\ \mathcal{M}_P, x \Vdash A \text{ iff } \mathcal{M}_N, x \Vdash A$$

We proceed by induction on the weight of A. Again, for the base case, A atomic it holds by definition as $[\![\ ]\!]$ is the same in the two models. For the propositional cases statement (b) easily follows by inductive hypothesis. We only consider the case $A = Bel_i(C|B)$. As in previous theorem, we use the following abbreviations: $u \Vdash_P B$ instead of $\mathcal{M}_P, u \Vdash B$ and $u \Vdash_N B$ instead of $\mathcal{M}_N, u \Vdash B$.
[$\Rightarrow$] Suppose that $x \Vdash_P Bel_i(C|B)$. This means that

Either for all $y \sim_i x$, $y \Vdash_P \neg B$ or there exists a $w \sim_i x$ and $w \Vdash_P B$ and for all $z \preceq_i w$, $z \Vdash_P B \to C$.

Suppose first that for all $y \sim_i x$, $y \Vdash_P \neg B$. Take any $\alpha \in N_i(x)$. By definition, $\alpha = \downarrow^{\preceq_i} z$, for some $z \sim_i x$. Let $y \in \downarrow^{\preceq_i} z$. Then by definition $y \preceq_i z$ and by plausibility implies possibility, $y \sim_i z$; thus by transitivity $y \sim_i x$. By hypothesis we have $y \Vdash_P \neg B$, whence by inductive hypothesis also $y \Vdash_N \neg B$. We showed $\alpha \Vdash^\forall \neg B$ for any $\alpha \in N_i(x)$, thus $x \Vdash_N Bel_i(C|B)$ holds (first case of the truth condition).
Suppose now that there is a $w \sim_i x$ such that $w \Vdash_P B$ and for all $z \preceq_i w$, $z \Vdash_P B \to C$. Let us consider $\alpha = \downarrow^{\preceq_i} w$. By inductive hypothesis $w \Vdash_N B$ and since $w \in \downarrow^{\preceq_i} w$ we obtain $\alpha \Vdash^\exists B$. Now consider any $u \in \alpha = \downarrow^{\preceq_i} w$. By definition $u \preceq_i w$. Thus by hypothesis $u \Vdash_P B \to C$, whence by inductive hypothesis also $u \Vdash_N B \to C$. We showed that $\alpha \Vdash^\forall B \to C$.

[$\Leftarrow$] Suppose that $x \Vdash_N Bel_i(C|B)$, this means that

Either for all $\alpha \in N_i(x)$ it holds that $\alpha \Vdash^\forall \neg B$ or there exists $\beta \in N_i(x)$ such that $\beta \Vdash^\exists$ and $\beta \Vdash^\forall B \to C$.

Suppose for all $\alpha \in N_i(x)$, $\alpha \Vdash^\forall \neg B$ holds. Let $y \sim_i x$, we want to show that $y \Vdash_P \neg B$. Since $y \sim_i x$, we have $\downarrow^{\preceq_i} y \in N_i(x)$. Thus by hypothesis $\downarrow^{\preceq_i} y \Vdash^\forall \neg B$ and $y \Vdash_N \neg B$, whence by inductive hypothesis also $y \Vdash_P \neg B$.

Suppose that there exists $\beta \in N_i(x)$ such that $\beta \Vdash^\exists B$ and $\beta \Vdash^\forall B \to C$. We prove that for some $u \sim_i x$ we have $u \Vdash_P B$ and for all $v \preceq_i u$ it holds that $v \Vdash_P B \to C$. By definition $\beta = \downarrow^{\preceq_i} z$ for some $z \sim_i x$. Since by hypothesis $\beta \Vdash^\exists B$, there exists $u \in \beta$ such that $u \Vdash_N B$, whence also $u \Vdash_P B$ by inductive hypothesis. By definition of β, we have $u \preceq_i z$ and thus $u \sim_i x$. Let now $v \preceq_i u$. By transitivity $v \in \beta$, and since $\beta \Vdash^\forall B \to C$ we have $v \Vdash_N B \to C$, whence also $v \Vdash_P B \to C$ by inductive hypothesis.

(End of the proof). We proved that if A is valid in $\mathcal{M}_N$ then A is also valid in $\mathcal{M}_P$. Suppose that A is valid in $\mathcal{M}_N$. Thus, for all $w \in W$, we have $w \Vdash_N A$, and by (b) we have also $w \Vdash_P A$ for all $w \in W$, which means that A is valid in $\mathcal{M}_P$. Finally, let A be valid in the class of N-models. Then, A is also valid in the class of *EP*-models. Given a *EP*-model $\mathcal{M}_P$, we build an N-model $\mathcal{M}_N$ as above. By hypothesis A is valid in $\mathcal{M}_N$ and for what we have just shown A is valid in $\mathcal{M}_P$. This concludes the proof. $\square$

Putting the two previous theorems together and making use of Theorem 6.3.1 we finally obtain the following:

Theorem 6.6.4. A formula A is a theorem of $\mathbb{CDL}$ if and only if it is valid in the class of plausibility models.

6.7 Other epistemic and doxastic modalities

Following [13] and [83], we add to $\mathbb{CDL}$ the doxastic operators of *safe belief* and *strong belief*. These operators can be defined both in terms of epistemic plausibility models and in terms of neighbourhood models. Starting from the neighbourhood models characterization, we give sequent calculus rules for these operators and extend the sequent calculus **G3CDL** to cover these modalities. Similarly, we define in both models a modal operator $[>]_i$, that expresses a strict order relation allowing to define two additional modalities: *weakly safe belief* and the *operator of unary revision.*

The safe belief operator captures the epistemic attitude corresponding to "Stalnaker's knowledge": according to Stalnaker, knowledge is a doxastic attitude which remains stable in front of belief revision with any *true* information [13]. Hintikka, following [62] made explicit the view that (a strong notion of) knowledge should be defined as a belief stable under the acquisition of further information.

> If someone says "I know that p" in this strong sense of knowledge, he implicitly denies that any further information would have led him to alter his view. He commits himself to the view that he would still persist in saying that p is true (...) even if he knew more than he now knows. [43, pp. 20–21]

Following [13] we use the term "knowledge" for the modality K_i, and call the present attitude of undefeasible knowledge "safe belief". The intuitive meaning of the safe belief operator $Bel_i^{\text{Safe}}A$ is that agent i safely believes A if and only if A is true, she believes A, and she continues to believe A whatever *true* information is received.

In terms of epistemic plausibility models, the safe belief operator is defined as follows [13, 83]:

$(Safe_P)$ $\quad \mathcal{M}_P, x \Vdash Bel_i^{\text{Safe}}A$ *iff for all* $y \preceq_i x$, $\mathcal{M}_P, y \Vdash A$.

We give the following condition in terms of neighbourhood models:

$(Safe_N)$ $\mathcal{M}_N, x \Vdash Bel_i^{\text{Safe}}A$ *iff there exists* $\alpha \in N_i(x)$ *such that* $x \in \alpha$ *and* $\alpha \Vdash^{\forall} A$.

To prove that the two notions match, we have to extend the inductive proofs of Theorems 6.6.2 and 6.6.3 on the equivalence between epistemic plausibility models and neighbourhood models. More precisely, we have to add a suitable inductive step which takes into account also the strong belief operator. The key fact is expressed in the next proposition.

Proposition 6.7.1. The extension of preferential models by the truth condition for the safe belief operator, $Safe_P$, gives the same class of valid formulas as the extension of neighbourhood models with condition $Safe_N$.

Proof. Let $\mathcal{M}_P$ be an epistemic plausibility model. We construct a neighbourhood model as in the proof of Theorem 6.6.2. We now have to prove that

$(a+)$ $\mathcal{M}_P, x \Vdash Bel_i^{\text{Safe}}A$ iff $\mathcal{M}_N, x \Vdash Bel_i^{\text{Safe}}A$

from the assumption that $[\![A]\!]^{\mathcal{M}_N} = [\![A]\!]^{\mathcal{M}_P}$. In order to prove the left-to-right direction, suppose $\mathcal{M}_P, x \Vdash Bel_i^{\text{Safe}} A$, i.e. for all $y \preceq_i x$, $y \Vdash A$. This means that for all $y \in \downarrow^{\preceq_i x}$, $y \Vdash A$, i.e. $\downarrow^{\preceq_i} x \Vdash^{\forall} A$. By construction we have $\downarrow^{\preceq_i} x \in N_i(x)$, and therefore there exists $\alpha \in N_i(x)$ such that $x \in \alpha$ and $\alpha \Vdash^{\forall} A$, i.e. $\mathcal{M}_N, x \Vdash Bel_i^{\text{Safe}} A$. As for the other direction of $(a+)$, suppose that $\mathcal{M}_N, x \Vdash Bel_i^{\text{Safe}} A$. This means there exists $x \in N_i(x)$ such that $x \in \alpha$ and $\alpha \Vdash^{\forall} A$. By construction, $\alpha = \downarrow^{\preceq_i} z$ for some z, $z \sim_i x$. Since $x \in \alpha$, then $x \in \downarrow^{\preceq_i} z$. This implies that $\downarrow^{\preceq_i} x \subseteq \downarrow^{\preceq_i} z$, and since $\downarrow^{\preceq_i} z \Vdash^{\forall} A$, we have a fortiori $\downarrow^{\preceq_i} x \Vdash^{\forall} A$, i.e., for all $y \preceq_i x$, $y \Vdash A$.

For the other direction of the proposition, let $\mathcal{M}_N$ be a neighbourhood model. We construct from it a plausibility model $\mathcal{M}_P$ following the procedure described in the proof of Theorem 6.6.3. We now have to prove that

(b+) $\mathcal{M}_N, x \Vdash Bel_i^{\text{Safe}} A$ iff $\mathcal{M}_P, x \Vdash Bel_i^{\text{Safe}} A$

assuming as hypothesis that $[\![A]\!]^{\mathcal{M}_N} = [\![A]\!]^{\mathcal{M}_P}$. For one direction, suppose that $\mathcal{M}_P, x \Vdash Bel_i^{\text{Safe}} A$. This means that for all $y \preceq_i x$, $\mathcal{M}_P, y \Vdash A$), i.e. from the definition of $\mathcal{M}_P$:

(*hp1*) for all $y \in W$, for all $\forall \beta \in N_i(x)$ if it holds that if $x \in \beta$ then $y \in \beta$, then $\mathcal{M}_N, y \Vdash A$

We have to prove that there exists $\alpha \in N_i(x)$ such that $\in \alpha$ and $\alpha \Vdash^{\forall} A$. We proceed by absurdum, assuming as hypothesis the negation of $(*)$:

(*hp2*) for all $\alpha \in N_i(x)$ it holds that if $x \in \alpha$ then $\alpha \nVdash^{\forall} A$

Let $\Sigma = \{\alpha \in N_i(x) \,|\, x \in \alpha\}$ (i.e. Σ is the principal filter generated by x in $N_i(x)$). By total reflexivity, we have that $\Sigma \neq \emptyset$. Let $\alpha^* = \bigcap \Sigma$. By the intersection property we have that $\alpha^* \neq \emptyset$, and by strong intersection property we have that $\alpha^* \in N_i(x)$ (and that $\alpha^* \in \Sigma$ as well). Thus we have that $x \in \alpha^*$, and it holds that for all $\beta \in N_i(x)$, $\alpha^* \subseteq \beta$. By (*hp2*) we conclude $\alpha^* \nVdash^{\forall} A$; thus, there exists $y \in \alpha^*$ such that $y \nVdash A$.

We now show that $y \preceq_i x$, in order to apply (*hp1*). Consider an arbitrary $\beta \in N_i(x)$ and suppose $x \in \beta$. Then $\alpha^* \subseteq \beta$ and, if $y \in \alpha^*$, $y \in \beta$, i.e. it holds that for all $\beta \in N_i(x)$, if $x \in \beta$ then $y \in \beta$. Apply (*hp1*) to conclude $y \Vdash A$ (for arbitrary y), in contraction with the fact that there exists $y \in \alpha^*$ such that $y \nVdash A$.

As for the other direction of $(b+)$, suppose that $\mathcal{M}_N \Vdash Bel_i^{\text{Safe}} A$. We have as hypothesis that there exists $\alpha \in N_i(x)$ such that $x \in \alpha$ and $\alpha \Vdash^{\forall} A$. We want to prove that for all $\beta \in N_i(x)$, if it holds that if $x \in \beta$ then $y \in \beta$, then $\mathcal{M}_N, y \Vdash A$. Given an arbitrary y, suppose that for all $\beta \in N_i(x)$ if $x \in \beta$ then $y \in \beta$; we have to show that $y \Vdash A$. By hypothesis there is an $\alpha_0 \in N_i(x)$ such that $x \in \alpha_0$ and $\alpha_0 \Vdash^{\forall} A$. Thus, since $x \in \alpha_0$, also $y \in \alpha_0$ (by hypothesis) and $y \Vdash A$. □

The notion of strong belief can be found in [99], where it is called "robust belief"; in more recent years, the notion was treated in [14], [13] and [83]. According to Baltag and Smets[5], the strong belief operator can be defined in terms of knowledge and safe belief:

$Bel_i^{\text{Strong}} A$ *iff* $Bel_i A \wedge K_i(A \rightarrow Bel_i^{\text{Safe}} A)$

Intuitively, a strong belief formula $Bel_i^{\text{Strong}} A$ says that an agent i strongly believes A if she believes A, and if she knows that if A is true, then she safely believes A,

[5]Pacuit provides a slightly different characterization of the operator, always in terms of epistemic plausibility models: $\mathcal{M}_P, x \Vdash Bel_i^{Strong} A$ *iff there exists* $y \sim_i x$ *such that* $y \Vdash A$, *and* $[\![A]\!] \cap [x]_{\sim_i} \preceq_i [\![\neg A]\!] \cap [x]_{\sim_i}$), *where for* $S, S' \subseteq W$, *let* $S \preceq_i S'$ *iff for all* $x \in S \forall y \in S'$ *it holds that* $x \preceq_i y$.

$$\frac{x \in a, a \in N_i(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x : Bel_i^{\text{Safe}} A, \Gamma, \Rightarrow \Delta} \text{ LSf (a!)}$$

$$\frac{a \in N_i(x), x \in a, \Gamma \Rightarrow \Delta, x : Bel_i^{\text{Safe}} A, a \Vdash^{\forall} A}{a \in N_i(x), x \in a, \Gamma \Rightarrow \Delta, x : Bel_i^{\text{Safe}} A} \text{ RSf}$$

$$\frac{a \in N_i(x), y \in a, y : A, \Gamma \Rightarrow \Delta, y : Bel_i^{\text{Safe}} A}{\Gamma \Rightarrow \Delta, x : K_i^{\text{Safe}} A} \text{ R}K_i^{\text{Safe}} \text{ (y, a!)}$$

$$\frac{a \in N_i(x), y \in a, x : K_i^{\text{Safe}} A, \Gamma \Rightarrow \Delta, y : A \quad y : Bel_i^{\text{Safe}} A, a \in N_i(x), y \in a, x : K_i^{\text{Safe}} A, \Gamma \Rightarrow \Delta}{a \in N_i(x), y \in a, x : K_i^{\text{Safe}} A, \Gamma \Rightarrow \Delta} \text{ L}LK_i^{\text{Safe}}$$

$$\frac{a \in N_i(x), a \Vdash^{\forall} A, x : K_i^{\text{Safe}} A, \Gamma \Rightarrow \Delta}{x : Bel_i^{\text{Strong}} A, \Gamma \Rightarrow \Delta} \text{ LSg (a!)}$$

$$\frac{a \in N_i(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A \quad a \in N_i(x), \Gamma \Rightarrow \Delta, x : K_i^{\text{Safe}} A}{a \in N_i(x), \Gamma \Rightarrow \Delta, x : Bel_i^{\text{Strong}} A} \text{ RSg}$$

Figure 6.4: Other epistemic and doxastic modalities

i.e. A is stable under belief revision with any true information. This condition can be expressed in terms of epistemic plausibility models. Recall first the truth condition for the unconditional belief operator in plausibility models in Observation 6.6.1: $\mathcal{M}, x \Vdash Bel_i A$ *iff there exists* $y \sim_i x$ *such that* $y \Vdash A$ *and for all* $z \preceq_i y$, $z \Vdash A$) [6]. We have:

$$\mathcal{M}_P, x \Vdash Bel_i^{Strong} A \textit{ iff } \quad \begin{array}{l} \textit{there exists } y \sim_i x \textit{ and for all} z \preceq_i y,\ z \Vdash A \textit{ and} \\ \textit{for all } z \sim_i x \textit{ if } z \Vdash A \textit{ then for all } y \preceq_i z,\ y \Vdash A. \end{array}$$

The condition can be translated in terms of neighbourhood models in an immediate way as follows:

$\mathcal{M}_N, x \Vdash Bel_i^{Strong} A$ *iff there exists* $\alpha \in N_i(x)$ *such that* $\alpha \Vdash^{\forall} A$ *and for all* $\beta \in N_i(x)$, *for all* $y \in \beta$ *if* $y \Vdash A$ *then there exists* $\gamma \in N_i(x)$ *such that* $y \in \gamma$ *and* $\gamma \Vdash^{\forall} A$.

The sequent calculus rules for both safe and strong belief can be derived from the definitions of the operators in terms of neighbourhood models. We factorize the complex semantic condition for strong belief by introducing an additional operator K_i^{Safe}, corresponding to the second conjunct of the above definition. The modality K_i^{Safe} could be interpreted by means of doxastic introspection: it means that if the agent knows that A is true, then she safely believes A. For the present scope, however, we employ this modality as a technical device.

$$x : K_i^{\text{Safe}} \equiv \begin{array}{l} \textit{for all } b \in N_i(x), \textit{ if for all } y \in b,\ y \Vdash A \textit{ then} \\ \textit{there exists } c \in N_i(x) \textit{ such that } y \in c \textit{ and } c \Vdash^{\forall} A. \end{array}$$

Note that the rules for strong belief introduce the simple and safe belief operators in the premisses, in accordance with the definition of the operator.

Observe that the characterisation of strong belief is guaranteed by the rules of the calculus, since it is easy to prove that for arbitrary x the following sequents are derivable

$$x : Bel_i^{\text{Strong}} A, \Gamma \Rightarrow \Delta x : Bel_i A \wedge K_i(A \rightarrow Bel_i^{\text{Safe}} A)$$

[6]Since the strong belief operator can be defined in terms of the other epistemic operators, we do not explicitly extend the theorem of equivalence between models.

$$x : Bel_i A \wedge K_i(A \rightarrow Bel_i^{\text{Safe}} A)\Gamma \Rightarrow \Delta x : Bel_i^{\text{Strong}} A$$

Baltag and Smets also consider the epistemic modality that expresses a strict order on plausibility models [13], i.e., the following operator:

$(>_{\mathsf{P}})$ $\mathcal{M}_P, x \Vdash [>]_i A$ *iff* *for all* $y <_i x$, $y \Vdash A$.

In terms of neighbourhood models, the definition of the $[>]_i$ operator is the following:

$(>_{\mathsf{N}})$ $\mathcal{M}_N, x \Vdash [>]_i A$ *iff* *for all* $\alpha \in N_i(x)$, *if* $x \notin \alpha$ *then* $\alpha \Vdash^{\forall} A$.

The $[>]_i$ operator is not particularly meaningful by itself; however, it can be used to define the operator of *weakly safe belief* and the (more interesting) operator of *unary revision*, respectively:

$$Bel_i^{Weak} A := A \wedge [>]_i A \qquad\qquad *_i\, A := A \wedge [>]_i \neg A$$

Observe that $x \Vdash Bel_i^{Weak} A$ holds only if x is a minimal world with to respect to the strict relation $<_i$, where for *minimal* is meant that all smaller worlds do not satisfy A. We now prove the equivalence of conditions $(>_{\mathsf{P}})$ and $(>_{\mathsf{N}})$, thus proving the equivalence of the two classes of models also with respect to this operator. Again, the proof is an extension of those of Theorems 6.6.2 and 6.6.3, as in the strong belief operator case.

Proposition 6.7.2. The definition of the safe belief operator in preferential models, expressed by condition $(>_{\mathsf{P}})$, is equivalent to the the definition of the operator in neighbourhood models, expressed by condition $(>_{\mathsf{N}})$.

Proof. Suppose we have a plausibility model $\mathcal{M}_P$. We build a neighbourhood model $\mathcal{M}_N$ as described in the proof of Theorem 6.6.2. We now have to prove the following proposition, assuming as hypothesis that $[\![A]\!]^{\mathcal{M}_N} = [\![A]\!]^{\mathcal{M}_P}$:

$(a++)$ $\mathcal{M}_N, x \Vdash [>]_i A$ *iff* $\mathcal{M}_P, x \Vdash [>]_i A$

In order to prove one direction, take as hypothesis that for all $\alpha \in N_i(x)$, if $x \notin \alpha$ then $\alpha \Vdash^{\forall} A$. We want to prove that for all $y <_i x$, $y \Vdash A$. Suppose $y <_i x$. Then, by construction, $x \notin \downarrow^{\preceq_i} y$, for $\downarrow^{\preceq_i} y = \{u \in W \mid u \preceq y\}$. We have that $\downarrow^{\preceq_i} y = \alpha$, for some α. By hypothesis, $\alpha \Vdash^{\forall} A$. Then, since $y \in \alpha$, we have that $y \Vdash A$.

As for the other direction, we assume as hypothesis that for all $y <_i x$, $y \Vdash A$. We want to prove that for all $\alpha \in N_i(x)$, if $x \notin \alpha$ then $\alpha \Vdash^{\forall} A$. Suppose $\alpha \in N_i(x)$ and $x \notin \alpha$. By construction, $\alpha = \downarrow^{\preceq_i} y$, for some $y \sim_i x$, and $x \notin \downarrow^{\preceq_i} y$. Thus, we have $y < x$. By hypothesis, $y \Vdash A$. Since this holds for all y such that $x \notin \downarrow^{\preceq_i} y$, and since $y \in \downarrow^{\preceq_i} y$, we have that $\downarrow^{\preceq_i} y \Vdash^{\forall} A$.

Suppose we have a neighbourhood model $\mathcal{M}_N$. We built a plausibility model $\mathcal{M}_P$ from it, as described in Theorem 6.6.3. In order to build the plausibility model, we will use the following additional condition:

$y \prec_i x$ iff (1) *for all* $\alpha \in N_i(x)$ *if* $x \in \alpha$ *then* $y \in \alpha$ *and*
(2) *there exists* $\beta \in N_i(x) = N_i(y)$ *such that* $y \in \beta$ *and* $x \notin \beta$.

We have to prove the following statement, always under the hypothesis $[\![A]\!]^{\mathcal{M}_N} = [\![A]\!]^{\mathcal{M}_P}$:

$(b++)$ $\mathcal{M}_N, x \Vdash [>]_i A$ *iff* $\mathcal{M}_P, x \Vdash [>]_i A$

To prove one direction of $(b++)$, suppose for all $\alpha \in N_i(x)$ if $x \notin \alpha$ then $\alpha \Vdash^{\forall} A$. We want to show that for all $y \prec_i x$, $y \Vdash A$. Suppose $y \prec_i x$. This means that (1) for all $\alpha \in N_i(x)$ if $x \in \alpha$then $y \in \alpha$ and (2) there exists $\beta \in N_i(x) = N_i(y)$ such that $y \in \beta$ and $x \notin \beta$. Note that the condition of the equality of spheres in (2) is justified by the following reasoning: by total reflexivity, $y \in \alpha$, and by absoluteness we have $N_i(x) = N_i(y)$. From (2), we have that there exists a sphere β_0 such that $\beta_0 \in N_i(x)$, $y \in \beta_0$ and $x \notin \beta_0$. By hypothesis, we have that $\beta_0 \Vdash^{\forall} A$. Thus, since $y \in \beta_0$, $y \Vdash A$.

As for the other direction, assume that for all $y \prec_i x$, $y \Vdash A$. We want to show that for all $\alpha \in N_i(x)$ if $x \notin \alpha$ then $\alpha \Vdash^{\forall} A$. Let $\alpha \in N_i(x)$ such that $x \notin \alpha$, and let $u \in \alpha$; we have to show that $u \Vdash A$. Let $\Sigma = \{\gamma \,|\, u \in \beta \textit{ and } x \notin \gamma\}$. Since $\alpha \in \Sigma$, $\Sigma \neq \emptyset$. Let $\delta = \cap\Sigma$; by the strong intersection property, $\delta \in N_i(x)$ and $\delta \neq \emptyset$.

It holds that $u \in \delta$. We want to show that $u \prec_i x$, since by hypothesis this implies $u \Vdash A$. Thus, we have to prove that condition (1) and (2) hold. Condition (2) holds by construction; as for (1), let $\beta \in N_i(x)$ such that $x \in \beta$; we have to prove that also $u \in \beta$. By nesting, it holds that either $\beta \subseteq \delta$ or $\delta \subseteq \beta$. The case $\beta \subseteq \delta$ is not possible, since we have set that $x \in \beta$, but by construction we have that $x \notin \delta$. Thus, it must hold that $\delta \subseteq \beta$; since by construction $u \in \delta$, we have $\in \beta$. Thus, by hypothesis we can conclude $u \Vdash A$, and the proposition in proved. □

The following informal observation should be useful to get an idea of the motivation behind the definition in neighbourhood models of the operators we have introduced in this section. Let us consider a world x and the set $N_i(x)$ of neighbourhoods associated to it. We can split $N_i(x)$ into two sets, namely:

$$N_i(x)^+ = \{\alpha \in N_i(x) \,|\, x \in \alpha\} \qquad N_i(x)^- = \{\alpha \in N_i(x) \,|\, x \notin \alpha\}$$

These represent, respectively, the set of neighbourhoods to which x belongs and the set of neighbourhoods to which x does not belong. Now recall the four modalities which can be defined in a standard way in neighbourhood models:[7]

$$\begin{array}{lll}
x \Vdash \Box^{\forall} A & \textit{iff} & \textit{for all } \alpha \in N_i(x) \textit{ it holds that } \alpha \Vdash^{\forall} A \\
x \Vdash \Box^{\exists} A & \textit{iff} & \textit{there exists } \alpha \in N_i(x) \textit{ such that } \alpha \Vdash^{\forall} A \\
x \Vdash \Diamond^{\forall} A & \textit{iff} & \textit{for all } \alpha \in N_i(x) \textit{ it holds that } \alpha \Vdash^{\exists} A \\
x \Vdash \Diamond^{\exists} A & \textit{iff} & \textit{there exists } \alpha \in N_i(x) \textit{ such that} \alpha \Vdash^{\exists} A
\end{array}$$

Note that the simple belief operator $x \Vdash Bel_i A$ iff there exists $\alpha \in N_i(x)$ such that $\alpha \Vdash^{\forall} A$ corresponds to the $\Box^{\exists}$ modality, while the knowledge operator $x \Vdash K_i A$ iff for all $\alpha \in N_i(x)$ it holds thay $\alpha \Vdash^{\forall} A$ corresponds to the $\Box^{\forall}$ modality.

Furthermore, all the operators that we have taken into account in this section can be interpreted as one of the above modalities, defined either on $N_i(x)^+$ or $N_i(x)^-$. More precisely, the safe belief operator corresponds to the $\Box^{\exists}$ modality defined on $N_i(x)^+$; the strong belief operator is defined on the same set. The $[>]_i$ operator corresponds to the $\Box^{\forall}$ modality defined on $N_i(x)^-$. The weakly safe belief operator and the unary revision operator are defined on the same set.

This overview gives an idea of the wide variety of modal operators which is possible to define in neighbourhood models. Following [13], we have restricted our analysis to the operators that should be interesting from an epistemic viewpoint—in principle, however, there are many others.

[7] Such modalities are denoted by $[\,]$, $\langle\,]$, $[\,\rangle$, $\langle\,\rangle$ in [83]; their proof theory is studied through labelled sequent calculi based on neighbourhood semantics in [73].

6.8 To conclude

In this chapter we presented a class of neighbourhood models, a multi-agent version of Lewis' sphere models, for the logic $\mathbb{CDL}$ of doxastic conditional beliefs. On the basis of this semantics, we have developed a labelled sequent calculus **G3CDL** for the logic.

There are a number of issues which may be objects of further investigation. First, $\mathbb{CDL}$ is the "static" logic that underlies dynamic extensions by *doxastic actions* [13]. It should be worth studying whether and how the labelled calculus can be extended to deal also with the dynamic extensions.

Furthermore, adequacy of the axiomatization with respect to multi-agent neighbourhood models with total reflexivity and absoluteness confirms that $\mathbb{CDL}$ is a multi-agent version of conditional logic $\mathbb{VTA}$. In Chapter 4 we have presented two variants of an hypersequent calculus for $\mathbb{VTA}$: $\mathcal{SH}^i_{\mathbb{VTA}}$ and $\mathcal{HH}^i_{\mathbb{VTA}}$, of which the latter is specifically tailored to treat systems with absoluteness. By extending system $\mathcal{HH}^i_{\mathbb{VTA}}$ to its multi-agent version we could define an internal calculus for $\mathbb{CDL}$, possibly more efficient than **G3CDL**. The challenge would to extend the proof of completeness of $\mathcal{HH}^i_{\mathbb{VTA}}$ to cover the multi-agent case.

From a computational viewpoint, **G3CDL** is far from optimal. The exact complexity of $\mathbb{CDL}$ is not known; we conjecture its upper bound to be PSPACE, but further investigations are needed to confirm this result. Some optimizations on the search strategy of **G3CDL** are possible, in particular to reduce the number of labels generated in a derivation. In any case, the method of constructing a labelled proof system from a system of semantics allowed us to define a cut-free sequent calculus - even if not optimal - for a logic which was lacking one. In this respect labelled calculi are an extremely useful tool, allowing to extract a proof system with strong structural properties for logics for which only semantic methods are known.

Concluding remarks

An overall picture

The main contribution of this work is the presentation of several systems of sequent calculi, both labelled and internal, for the families of preferential and counterfactual conditional logics[8]. Following Stalnaker's and Lewis' approach, these logics are defined by adding to the language of classical propositional logic a two place modal operator: either the conditional operator $>$ or the comparative plausibility operator $\preccurlyeq$.

Labelled calculi for preferential logics are defined in Chapter 3. These calculi modularly capture the systems extending $\mathbb{PCL}$, there including counterfactual logics. They rely on neighbourhood models, a general system of semantics which uniformly represents all $\mathbb{PCL}$ extensions. Chapter 3 provides an (indirect) proof of completeness of neighbourhood semantics with respect to the axiomatizations of the logics (Section 3.2). Internal calculi (Chapter 4) are defined only for extensions of counterfactual logic $\mathbb{V}$. These calculi, which allow a direct formula interpretation, enrich the Gentzen's sequent calculus for classical propositional logic with additional structural connectives, similarly as what is done to define sequent calculi for modal logics. Three proof systems are introduced, each adapted to capture different classes of logics: one-level nested sequents, extended hypersequents and head hypersequents.

Chapter 5 presents the two directions of a translation between proof systems for logic $\mathbb{V}$. This equivalence result underlines the correlation between the labelled and the internal framework. Moreover, since labelled calculi display a direct link with neighbourhood models, the mapping makes explicit the relationship between internal nested sequents and neighbourhood semantics for Lewis' logics. More precisely, conditional blocks are shown to be related to neighbourhoods, each block encoding comparative plausibility formulas regarding a neighbourhood. Then, even if the equivalence is formally proved only for system $\mathbb{V}$, an intuitive correspondence can be perceived between the labelled frameworks and the nested calculi for extensions of $\mathbb{V}$, and can in principle be extended to hypersequents and head-hypersequents as well.

Finally, Chapter 6 contains an application of the proof-theoretic methods developed in the preceding chapters to a multi-agent and epistemic setting. A labelled sequent calculus is introduced for the logic of conditional belief ($\mathbb{CDL}$), which is a multi-agent version of the conditional logic $\mathbb{VTA}$. The semantics of the logic is defined in terms of multi-agent neighbourhood models, with (direct) proofs of soundness and completeness to support the operation. Then, the blueprint developed for conditional logics is applied to obtain a multi-agent labelled calculus for this highly expressive system of logic.

Many things are still missing from an ideally complete proof-theoretic perspective on conditional logics. First of all, while we gave a modular and (almost) exhaustive presentation of labelled sequent calculi for preferential and counterfactual logics, uniform

[8]Here and throughout the thesis by preferential conditional logics we mean logic $\mathbb{PCL}$ and all its extensions, and by counterfactual conditional logics, or Lewis' logics, we denote system $\mathbb{V}$ and its extensions. Adopting this terminology, Lewis' logics are a subfamily of preferential logics.

and internal framework for the same systems are still missing. Possibly one of the most difficult open task is to complete the cube of internal calculi for conditional systems. The problem is that, when compared to labelled systems, internal rules are much harder to define. With labelled calculi it is usually possible to individuate a general blueprint which guides the rules definition[9]. As a consequence, labelled calculi are highly versatile, and can be employed to define calculi for logics for which no sequent calculus is known, as in the case of $\mathbb{PCL}$. Conversely, internal proof systems usually need to be defined from scratch. Compare, for instance, the sequent calculi for modal logics presented in Chapter 2 with the proof systems for Lewis' logics introduced in Chapter 4. The structures of conditional blocks and transfer blocks defined for Lewis' logics are system-specific: they cannot be retrieved from proof systems for modal logics, and it is unlikely that they can be employed in calculi for other logics. In particular, they cannot be adapted to define internal calculi for $\mathbb{PCL}$ and its extensions: without the condition of nesting, the truth condition for $\preccurlyeq$ becomes much more complex, and can no longer be treated by means of the conditional block structure.

Internal calculi would be needed to implement automated theorem provers for conditional logics. For most preferential and counterfactual logics such tools are at present missing, as is it missing a thorough comparative study on which internal proof system could be better suited to implement automated reasoning. From a theoretical perspective, internal calculi could be exploited as a basis for investigating proof-theoretical properties of the logics, as the strongly analytic property of interpolation. Then, another theoretical application would be the definition of uniform translations between labelled and internal frameworks. Other than clarify the link of internal calculi with the relevant class of models, the translation could allow an almost direct countermodel construction from failed proof search, which is less immediate to define in the case of internal calculi.

Finally, conditional logics should be placed in a wider perspective, by investigating the relationships between these systems and other modal logics from an axiomatic, a model-theoretic and a proof-theoretic viewpoint. By means of a semantic reasoning, Lewis identified the *outer* modal logics corresponding to counterfactual systems (Section 1.6). Proof-theoretically, internal calculi for the outer modal logics could be retrieved from calculi for conditional logics. As another example, the relationship between preferential logics and non-monotonic systems is known from [47]. Moreover, the axiomatic and semantic correspondence between conditional logic $\mathbb{VTA}$ and the epistemic logics $\mathbb{CDL}$ (Chapter 6) could be extended to other systems. In particular, the relationship of preferential and counterfactual frameworks with conditional deontic logics has not yet been studied. These systems, introduced by van Fraassen [60], are an extension of von Wright modal deontic logic [106] with a two-places modal operator expressing conditional obligation. They allow to solve various paradoxes and contrary-to-duty imperatives. Conditional deontic systems can be further extended: for instance, defeasible deontic logics incorporate a mechanism to establish preference hierarchies of norms, solving problems of deontic conflicts[10]. Most likely, these logics are overlapping with conditional logics, but it is not clear in which measure. The proof theory of conditional logics could be adapted to similar, or equivalent, formal systems.

[9]Usually, such a blueprint consists in a method of importing into the language of the calculus relevant semantic elements. Refer to [69, 23] for the methodology to be applied in case of Kripke semantics, and to [72] in the case of neighbourhood models.

[10]There is a vast bibliography on the subject; refer to [95] for an overview.

Future work

This work leaves open several research directions, which we here consider in detail. For the sake of clarity, we shortly address first proof-theoretic open issues, and then list the model-theoretic research paths.

Regarding the calculi, possibly the most important open problem is the definition of internal and standard calculi for $\mathbb{PCL}$ and its extensions, excluding Lewis' systems. Moreover, the calculi in Chapter 4 do not cover the full extent of Lewis' cube of counterfactual logics. In particular, complete calculi for $\mathbb{VA}$ and $\mathbb{VNA}$ and calculi for $\mathbb{VU}$ and $\mathbb{VNU}$ are missing. We conjecture that calculi for all these systems can be given in terms of *grafted hypersequents*. This formalism was introduced by Kuznets and Lellman to define proof systems for modal logic **K5** and **K45** [51]. With respect to **S5**, these logics are missing reflexivity; as a consequence, the hypersequent formula interpretation is not suitable for them, since the actual world could be "different" from the other worlds. The solution of grafted hypersequents consists in reserving the first component of the hypersequent to represent formulas of the actual world, and defining two different sets of rules: a set to be applied to the first component, and a set to be applied to all other components. Since the outer modal logics of both $\mathbb{VU}$ and $\mathbb{VA}$ is **K45**, and the outer modal logic of $\mathbb{VNU}$ and $\mathbb{VNA}$ is **KD45**, it should be possible to extend the grafted hypersequent framework to define calculi for these logics.

As for the internal calculi for Lewis' logic, it would be interesting to enrich the structure of sequents and give a fully nested version of the sequent calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ for $\mathbb{V}$ presented in Chapter 4. We defined $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ as a "nested" calculus, since the conditional blocks are similar to the block structure introduced by nested calculi. However, the level of nesting allowed by $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ is at most one, since conditional blocks cannot occur inside other conditional blocks. Introducing the block structure of nested calculi would allow for a real nesting of sequents, with the advantage that the calculus would be fully invertible, with rule jump becoming the invertible rule $\mathsf{jump}^{\mathsf{N}}$.

$$\frac{C \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd C]}\ \mathsf{jump} \qquad\qquad \frac{G\{\Gamma \Rightarrow \Delta, [C \Rightarrow \Sigma]\}}{G\{\Gamma \Rightarrow \Delta, [\Sigma \lhd C]\}}\ \mathsf{jump}^{\mathsf{N}}$$

The definition of such a calculus is object of our current research. Moreover, a nested system could probably offer a more direct mapping with the labelled proof systems for conditional logic with respect to the procedure presented in Chapter 5.

With respect to the equivalence result between sequent calculi for $\mathbb{V}$, implementing the mapping from internal to labelled proof systems is definitely worthy of further study, since this would give us "for free" the first implementation of labelled calculi, as mentioned at the end of Chapter 5.

An open problem concerning the proof theory of the logic of conditional belief concerns the definition of calculus for dynamic $\mathbb{CDL}$. More precisely, it should be investigated whether and how the labelled calculus defined in Chapter 6 for the static logic of conditional belief could be extended to a dynamic system in which the set of belief of the agent may be updated with new propositions. Moreover, we wondered whether we could define an internal calculus for the logic of conditional belief. The answer is affirmative - it suffices to extend the hypersequent calculus for $\mathbb{VTA}$ presented in Chapter 3 to its multi-agent version. The operation is non-trivial: it is necessary to resort to a nested calculus, since by indexing conditional blocks with agents, the hypersequent structure does no longer suffice. By means of example, this is how the rule of jump would look like in the nested multi-agent version of $\mathcal{SH}^{i}_{\mathbb{VTA}}$, for i label for agent:

$$\frac{G\{\Gamma \Rightarrow \Delta, [C \Rightarrow \Sigma]^{i}\}}{G\{\Gamma \Rightarrow \Delta, [\Sigma \lhd C]^{i}\}}\ \mathsf{jump}^{\mathsf{VTA}}$$

Completeness of this calculus is under scrutiny.

As for model-theoretic problems requiring further investigation, the most important one is probably the direct proof of completeness of the axiomatization of $\mathbb{PCL}$ with respect to neighbourhood models. Generally speaking, proving completeness of $\mathbb{PCL}$ seems to be a difficult task; moreover, the only proofs of syntactic completeness that can be found in the literature are relative to preferential models. A completeness proof with respect to neighbourhood models would additionally motivate the use of this class of models for the logics.

A promising research direction concerns neighbourhood models for multi-agent epistemic logics. It should be possible to extend to other epistemic operators the definition in terms of neighbourhood models, and consequently to define sequent rules for them, as discussed in Section 6.7. It would also be worth investigating the full extent of neighbourhood semantics by means of the analysis proposed at the end of the same section. The general modal operators presented there could be used to formalize in an uniform way other modal operators, not necessarily expressing epistemic or doxastic attitudes, and not necessarily belonging to the realm of normal modal logics. This will be object of future researches.

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