MOILab: towards a labelled theorem prover for intuitionistic modal logics

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Abstract

We present MOILab, a prototype Prolog theorem prover implementing a labelled sequent calculus for IK, the basic system in the intuitionistic modal logics family. MOILab builds upon MOIN, a theorem prover implementing nested sequent calculi (both single-conclusion and multi-conclusion) for all the logics in the modal intuitionistic cube. With respect to the nested implementations, MOILab offers a straightforward countermodel construction in case of proof search failure.

1 Introduction

We tackle the problem of defining automated theorem provers for intuitionistic modal logics. In [3] is introduced a SWI Prolog theorem prover for classical and intuitionistic modal logics, called MOIN[1]. The prover implements nested proof systems, as other Prolog provers do (refer to [6, 5] for a Prolog implementation of nested sequents for non-normal modal logics, and to [9] for normal conditional logics). MOIN implements nested sequent from [1] for classical modal logics and, for the 15 logics in the intuitionistic modal cube, it implements both single-conclusion (or Gentzen-style) nested sequents form [10, 8, 2] and multi-conclusion (or Maehara-style) nested sequents from [4]. For the systems whose decidability is known, the prover terminates.

We here present a prototype Prolog prover extending MOIN and implementing a labelled sequent calculus for IK, the basic system of intuitionistic modal logics. The prover is called MOILab, for *MOdal and Intuitionistic Labelled sequents*[2]. The labelled proof system, introduced in [7], internalises the semantic information from bi-relational models for intuitionistic modal logics into the sequent calculus syntax.

With respect to the nested systems for intuitionistic modal logics, the labelled calculus offers two main advantages: all its rules are invertible, and a countermodel can be easily extracted from the upper sequent of a failed branch. This motivates the introduction of MOILab. As to now, the theorem prover is a prototype: only the basic logic IK is implemented, and proof search might not terminate on some sequents.

2 Intuitionistic modal logic IK and labelled sequents

The language of intuitionistic modal logics extends the language of intuitionistic propositional logic with the modal operators $\Box$ and $\Diamond$. Lacking the De Morgan duality, there are several variants of the *distributivity axiom* that are classically but not intuitionistically equivalent. An

[1] MOIN stands for *MOdal and Intuitionistic Nested sequents*. The prover is available here: http://www.lix.polytechnique.fr/Labo/Lutz.Strassburger/Software/Moin/MoinProver.html

[2] MOILab is available here: http://mariannagirlando.com/MOILab.html

intuitionistic variant of modal logic K, called IK, is obtained by adding to an axiomatization of intuitionistic propositional logic the *necessitation rule* of K and the following axioms[3]:

$$\begin{array}{lll} \mathsf{k}_1: \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathsf{k}_3: \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) & \mathsf{k}_5: \Diamond\bot \supset \bot \\ \mathsf{k}_2: \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathsf{k}_4: (\Diamond A \supset \Box B) \supset \Box(A \supset B) & \end{array}$$

Bi-relational models for IK are defined by adding a valuation for atomic formulas to a bi-relational frame (refer to [7] for details):

Definition 2.1. A *bi-relational frame* $\mathcal{F}$ is a triple $\langle W, R, \leq \rangle$ of a set of worlds W equipped with an accessibility relation R and a preorder $\leq$ (*i.e.* a reflexive and transitive relation) satisfying:

(F_1) For all $u, v, v' \in W$, if uRv and $v \leq v'$, there exists u' s.t. $u \leq u'$ and $u'Rv'$.

(F_2) For all $u, u', v \in W$, if uRv and $u \leq u'$, there exists v' s.t. $u'Rv'$ and $v \leq v'$.

Reflecting this definition, the sequents of the labelled calculus $\mathsf{labIK}_{\leq}$ defined in [7] are equipped with two relation symbols, one for R and one for $\leq$.

Definition 2.2. A two-sided intuitionistic *labelled sequent* is of the form $\mathcal{R}, \Gamma \Longrightarrow \Delta$ where $\mathcal{R}$ denotes a set of relational atoms xRy and preorder atoms $x \leq y$, and Γ and Δ are multi-sets of labelled formulas $x{:}A$ (for x and y variables for labels and A intuitionistic modal formula).

Refer to [7] for a presentation of the sequent calculus rules. Moreover, in [7] it is proved that all $\mathsf{labIK}_{\leq}$ rules are invertible (Lemma 6.4), and that the cut rule is admissible (Theorem 6.1). The rules needed to extend $\mathsf{labIK}_{\leq}$ to logics whose axiomatization extends IK by *one-sided Scott-Lemmon axioms* are also defined.

3 Towards a labelled theorem prover

MOILab is composed of a set clauses, each implementing a rule of the labelled sequent calculus. The only exception is the reflexivity rule, which is applied together with the rules introducing (backwards) a new label. More details on the syntax and basic usage of MOILab can be found in the `readme` file. In case of proof search success, MOILab produces in output a derivation; otherwise, a countermodel is extracted from the upper sequent of a failed branch.

Thanks to invertibility of the $\mathsf{labIK}_{\leq}$ rules, backtrack points do not need to be introduced in proof search. The countermodel extraction is straightforward, since no information is lost in going from the conclusion to the premiss(es) of the rules.

Termination is still an issue: as for now, MOILab implements the naive strategy of not applying a rule to a labelled formula if the labelled formula to be introduced already occurs in the sequent. This is not enough to ensure termination of proof search, and a more refined termination strategy is currently under study. Once termination is settled, we plan to modularly extend MOILab to cover the other logics in the intuitionistic modal logics cube, using the rules defined in [7].

References

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[3] We employ the coloured syntax from [7]: variables for labels are blue and formulas are green. The aim is to improve readability.

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